

# Fixed-point structure of low-dimensional relativistic fermion field theories: Universality classes and emergent symmetry

Friedrich Gehring,<sup>1,\*</sup> Holger Gies,<sup>1,†</sup> and Lukas Janssen<sup>2,‡</sup>

<sup>1</sup>*Theoretisch-Physikalisches Institut, Abbe Center of Photonics,  
Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, 07743 Jena, Germany*

<sup>2</sup>*Department of Physics, Simon Fraser University, Burnaby, British Columbia V5A 1S6, Canada*

We investigate a class of relativistic fermion theories in  $2 < d < 4$  space-time dimensions with continuous chiral  $U(N_f) \times U(N_f)$  symmetry. This includes a number of well-studied models, e.g., of Gross-Neveu and Thirring type, in a unified framework. Within the limit of pointlike interactions, the RG flow of couplings reveals a network of interacting fixed points, each of which defines a universality class. A subset of fixed points are “critical fixed points” with one RG relevant direction being candidates for critical points of second-order phase transitions. Identifying invariant hyperplanes of the RG flow and classifying their attractive/repulsive properties, we find evidence for emergent higher chiral symmetries as a function of  $N_f$ . For the case of the Thirring model, we discover a new critical flavor number that separates the RG stable large- $N_f$  regime from an intermediate- $N_f$  regime in which symmetry-breaking perturbations become RG relevant. This new critical flavor number has to be distinguished from the chiral-critical flavor number, below which the Thirring model is expected to allow spontaneous chiral symmetry breaking, and its existence offers a resolution to the discrepancy between previous results obtained in the continuum and the lattice Thirring models. Moreover, we find indications for a new feature of universality: details of the critical behavior can depend on additional “spectator symmetries” that remain intact across the phase transition. Implications for the physics of interacting fermions on the honeycomb lattice, for which our theory space provides a simple model, are given.

## I. INTRODUCTION

The universal behavior of matter near the points of continuous phase transitions constitutes one of the most intriguing phenomena in statistical physics. Explaining universality was the great success of renormalization group (RG) theory, which has by now become one of our basic tools to understand systems with many interacting degrees of freedom. Systems near criticality can fall into universality classes which are characterized by only a few basic properties, independent of the microscopic interactions between the particles. In simple bosonic systems, the general characteristics that determine the universality class are well established: the system’s dimensionality, the symmetry of the order parameter, and the presence or absence of sufficiently long-ranged interactions. The reason for this simplicity for a large class of purely bosonic theories lies in the existence of just one critical RG fixed point, determining the critical behavior of all these theories within the corresponding theory space [1].

In systems with fermions, however, the situation can be more complex, and the question of the defining properties of the universality classes could be more subtle. Some issues that arise in these systems are the following: Is the above list exhaustive or can one find two systems with the same dimensionality, symmetry of the order parameter, and range of interaction, but *different* critical behavior?

Does, for instance, the critical behavior depend on additional (“spectator”) symmetries that do not take part in the symmetry breaking pattern? Can there be more than one critical RG fixed point in the same theory space? In fermionic systems, different types of interactions often lead to various possible pairing mechanisms, which allow excitations with new quantum numbers, and—upon condensation—lead to new collective modes. These new modes can be scalar fields as in the BCS theory of superconductivity [2], but other types such as vector modes are equally well possible [3]. Furthermore, fermionic self-interactions are inherently related to Fierz identities, with the help of which we can always rewrite any four-fermion term as a linear combination of a different set of four-fermion interactions. Focusing on only one particular interaction channel and neglecting all others within a single-channel approximation, as is usually done in mean-field approaches, involves an ambiguity which may substantially affect the validity of the approximation [3, 4]. In a RG approach, instead, one should incorporate *all* interaction channels that are invariant under the symmetry of a given system, and let the dynamics decide which one becomes dominant. At the same time, this allows to study a whole class of theories in a particular theory space and to investigate their decomposition into universality classes.

In this paper, we present a class of relativistic fermion field theories in  $2 < d < 4$  space-time dimensions, which allows to study these and related questions. In particular, we will investigate the space of theories with Lorentz, continuous chiral  $U(N_f) \times U(N_f)$ , and a set of discrete symmetries, with  $N_f$  being the number of four-component fermion flavors. This includes the ubiqui-

---

\* [friedrich.gehring@uni-jena.de](mailto:friedrich.gehring@uni-jena.de)

† [holger.gies@uni-jena.de](mailto:holger.gies@uni-jena.de)

‡ [lukasj@sfu.ca](mailto:lukasj@sfu.ca)

tous 3d Gross-Neveu models [5–13], as well as the 3d Thirring model [3, 14–20], both of which have been used as testing grounds to study nonperturbative phenomena in strongly-coupled fermion field theories, such as chiral symmetry breaking and nonperturbative renormalizability. Lately, these systems receive revived attention as effective models describing the physics of condensed matter systems that incorporate fermionic excitations with relativistic dispersion relation, in particular graphene [21–25] and the surface states of topological insulators [26]. Since interactions in graphene are strong [27], the question of possible quantum transitions from the semimetallic into different Mott insulating [21, 28] or superconducting [29] phases has extensively been investigated previously. We here take a somewhat different viewpoint: Instead of focusing on the various possible infrared (IR) phases [30], we target at the ultraviolet (UV) structure of our effective-theory space. With the aid of functional RG techniques, we map out the fixed points and the accompanying relevant and irrelevant directions. A fixed point with exactly one relevant direction corresponds to a possible second-order phase transition whose critical behavior it governs. We demonstrate that in the present theory space multiple such critical fixed points may exist, each one of them defining its own universality class. Different theories with identical field content and microscopic symmetries may therefore be in the domain of attraction of *different* critical fixed points and thus be in different universality classes—even though symmetry-breaking patterns and critical degrees of freedom may be completely the same.

The theory space we propose also represents a simple example to study the possibility of dynamical emergence of symmetry due to an IR-attractive RG fixed point in a higher-symmetric subspace. Emergent symmetry is a well-known phenomenon in various condensed-matter systems, which often exhibit rotational symmetry at low energy, while the microscopic Hamiltonian explicitly breaks (continuous) rotational invariance. For lattice regularizations of rotational-invariant field theories, dynamic enhancement of the rotation symmetry is essential in order to avoid the need for fine tuning. The possibility that Lorentz symmetry, instead of at very high energies being explicitly realized, could be emergent as a low-energy phenomenon is an old idea [31] and a key requirement for Horava’s approach to regularize quantum gravity [32]. It is also believed that the critical points in the graphene system are Lorentz symmetric [21, 22]. Close to a strong-coupling fixed point in (3+1)-dimensional conformal field theory [33] and in (2+1)-dimensional [34] and (3+1)-dimensional [35] fermionic lattice models supersymmetry could be emergent. Emergent supersymmetry would be highly desirable for lattice formulations of supersymmetric theories, which inevitably break at least part of the supersymmetry on the microscopic level [36]. The emergence of enhanced internal symmetries has also been observed near a fermionic multicritical point with  $\mathbb{Z}_2 \times O(2)$  symmetry [37] and in bosonic  $O(N_1) + O(N_2)$

models [38], leading to the seemingly paradoxical possibility that Goldstone modes can arise in models with initially only discrete symmetries.

The  $U(N_f) \times U(N_f)$  theory space we consider in this work includes the higher-symmetric  $U(2N_f)$  subspace, which is the theory space of the continuum Thirring model [3, 15]. Already for this simplistic example, the question whether or not lower-symmetric perturbations out of this subspace are relevant in the sense of the RG appears to be nontrivial. It turns out that it can actually depend on the number of flavors  $N_f$ : Within our approximation we find that for  $N_f > 6$  the  $U(2N_f)$ -symmetric subspace is IR attractive, in accordance with the large- $N_f$  analysis [16, 17], while it becomes IR repulsive for  $2 \leq N_f < 6$ . Due to an additional Fierz identity the  $N_f = 1$  case is special, and we again find that perturbations out of the  $U(2N_f)$  subspace are irrelevant, as in the large- $N_f$  case. These findings shed new light on previous simulation results that employ lattice formulations that generically break parts of the microscopic symmetries of the continuum theories [13, 16–19].

In order to determine the RG flow of our theory space we use Wetterich’s functional RG equation [39]. For a first analysis, we confine ourselves to the study of pointlike fermionic interactions which is similar in spirit to the quantitatively successful derivative expansion for bosonic theories. As we shall show, this approximation is equivalent to the usual Wilsonian one-loop RG and as such, will become exact to first order in  $d = 2 + \epsilon$  for all  $N_f$ , or in any dimension  $2 < d < 4$  for large  $N_f$ . For the physically interesting case of small  $N_f$  directly in  $d = 3$ , our simple approximation—as the great majority of all other analytical approaches in the nonperturbative domain—may be not sufficiently controlled. However, the use of the functional RG in the present case has important advantages: first, the method provides multiple systematic ways to straightforwardly improve the present simple approximation used here, e.g., by incorporating momentum-dependent vertices or by partial- or dynamical-bosonization techniques—all of which are employed and advanced in recent studies of similar systems [3, 40–44]. Second, our approach allows to derive a general formula for the one-loop flow of relativistic fermion models, which should be directly applicable to systems with an arbitrary number of interaction channels. We believe that this will be of relevance to future investigation of more complex systems with critical fermion interactions, e.g., in order to derive an effective theory for electrons in the single- [22] or bilayer [45] graphene in 2+1 dimensions, or in the quantum critical systems with quadratic band touching in 3+1 dimensions [46, 47].

The rest of the article is organized as follows: In the following section we define the theory space that we consider and discuss its symmetries. We briefly introduce our method in Sec. III which we will use in Sec. IV to derive a general formula for the one-loop beta function of relativistic fermion systems with pointlike four-fermion interactions. We discuss the flow equations for our sys-

tem in Sec. V. Section VI is devoted to general properties of the one-loop flow in four-fermion models. In Secs. VII and VIII we discuss the fixed-point structure for  $N_f = 1$  and  $N_f \geq 2$ , respectively. We give an outlook on possible phase transitions and critical behavior in Sec. IX, and conclude in Sec. X.

## II. FERMION MODELS WITH $U(N_f) \times U(N_f)$ SYMMETRY

We are interested in the theory space of the  $U(N_f) \times U(N_f)$ -symmetric Gross-Neveu model in  $2 < d < 4$  space-time dimensions, which may be defined by the microscopic action [12]

$$S_{\text{GN}} = \int d^d x \left[ \bar{\psi}^a i \gamma_\mu \partial_\mu \psi^a + \frac{\bar{g}}{2N_f} (\bar{\psi}^a \psi^a)^2 \right], \quad (1)$$

with space-time index  $\mu = 0, \dots, d-1$  and “flavor” index  $a = 1, \dots, N_f$ . Summation over repeated indices is implicitly understood. We use a four-dimensional representation of the Clifford algebra  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \mathbb{1}_4$ . The Dirac conjugate is given by  $\bar{\psi} = -i\psi^\dagger \gamma_0$ . A possible application of this model is the system of  $N_f$  species of fermions on the honeycomb lattice that interact via nearest-neighbor interactions. This interaction can be parametrized by the four-fermi coupling  $\bar{g}$  if the four components of the Dirac spinor  $\psi^a$  are associated with the electron annihilation operators on the two sublattices and the two Dirac points of the honeycomb system, with the “flavor”  $a$  representing the spin projection [21, 22]. Accordingly,  $N_f = 2$  for spin- $\frac{1}{2}$  particles. In  $d > 2$ , the coupling  $\bar{g}$  has positive mass dimension, reflecting the model’s perturbative nonrenormalizability. However, it is now well accepted that the existence of an UV stable fixed point ensures that in  $2 < d < 4$  the model is renormalizable *nonperturbatively* [5]—a fact that can be reinterpreted as maybe the simplest example for Weinberg’s asymptotic safety scenario [12]. In the honeycomb-lattice system the Gross-Neveu fixed point governs the phase transition into the charge density wave phase that is expected for large nearest-neighbor interaction [21]. The one-dimensional theory space defined by Eq. (1) is closed under the RG probably to any order [7, 8], i.e., once we start with a microscopic action of the form (1) no new interactions will be generated by RG transformations. This fact has been used to construct the fixed-point potential of the Gross-Neveu model at arbitrary order in the fermionic field [48]. However, as we shall see, the RG closedness of this model is a special property of the simple scalar-type interaction, and does generically not hold in systems with more complex interactions, such as in the Thirring model with a vector-type interaction [15]. We will also see that although the Gross-Neveu action defines a RG invariant subspace of theory space, infinitesimal perturbations on the microscopic level may, depending on the number of flavors  $N_f$ , be RG relevant and drive the system away from the simple Gross-Neveu theory.

In order to investigate the decomposition of the microscopic theories into universality classes—beyond just relying on the universality hypothesis—and to study the stability of the models with respect to perturbations, it is therefore mandatory to incorporate the RG flow of *all* operators that are invariant under the given symmetries. This defines our  $U(N_f) \times U(N_f)$  theory space: the space of all fermion theories which enjoy (at least) the symmetries of the Gross-Neveu model [Eq. (1)]. These are the following:

*Relativistic invariance:*

$$\psi(x) \mapsto e^{-\frac{i}{4} \epsilon_{\mu\nu} \gamma_{\mu\nu}} \psi(x'), \quad \bar{\psi}(x) \mapsto \bar{\psi}(x') e^{\frac{i}{4} \epsilon_{\mu\nu} \gamma_{\mu\nu}}, \quad (2)$$

where  $\gamma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]$  and  $\epsilon_{\mu\nu}$  is an antisymmetric tensor defining the rotation axis and angle in  $(2+1)$ -dimensional space-time. Here and in the following, where unambiguous, we suppress the flavor index  $a$ .  $x'_\mu = \Lambda_{\mu\nu}^{-1} x_\nu$  with  $\Lambda_{\mu\nu}$  being the generator of the space-time rotations. While the honeycomb lattice explicitly breaks Lorentz symmetry, it is, however, expected to dynamically emerge close to one of the critical points in graphene [21, 22]. Incidentally, the Lorentz transformation of Eq. (2) can be understood as a particular subgroup of the global spin-base transformations  $SL(4, \mathbb{C})$  [49] constituted by the similarity transformations of the Clifford algebra [50].

*Flavor symmetry:*

$$\psi^a \mapsto U^{ab} \psi^b, \quad \bar{\psi}^a \mapsto \bar{\psi}^b (U^\dagger)^{ba}, \quad (3)$$

with the unitary matrix  $U \in U(N_f)$ . Flavor symmetry is generated by the (generalized)  $N_f \times N_f$  Gell-Mann matrices  $\lambda_i$ ,  $i = 1, \dots, N_f^2 - 1$ , together with the identity  $\lambda_0 \equiv \mathbb{1}_{N_f}$ .

*Continuous chiral symmetry:* Due to our four-dimensional reducible representation of the Clifford algebra there now exist two additional Dirac matrices, which anticommute with all three  $\gamma_\mu$ :  $\gamma_3$  and  $\gamma_5$ . Their Hermitian product  $\gamma_{35} := i\gamma_3\gamma_5$  generates the continuous chiral symmetry<sup>1</sup>

$$\psi^a \mapsto e^{i\theta\gamma_{35}} \psi^a, \quad \bar{\psi}^a \mapsto \bar{\psi}^a e^{-i\theta\gamma_{35}}, \quad (4)$$

where  $\theta \equiv \theta(a)$  may depend on the flavor index  $a = 1, \dots, N_f$ . Continuous chiral symmetry and flavor symmetry together are therefore generated by the  $2N_f^2$  matrices  $\{\lambda_0, \dots, \lambda_{N_f^2-1}\} \otimes \{\mathbb{1}_4, \gamma_{35}\}$ , forming the global  $U(N_f) \times U(N_f)$  symmetry of the Gross-Neveu model in the reducible representation of the Clifford algebra. There is an alternative way to understand this symmetry [23]: By making use of the orthogonal projectors  $P_{L/R} := \frac{1}{2}(\mathbb{1} \pm \gamma_{35})$  with  $P_{L/R}^2 = P_{L/R}$ ,  $P_L P_R = 0$ , and

<sup>1</sup> Our Dirac matrix conventions are identical to those of, e.g., [3, 15] with the identification, i.e., simple renaming,  $\gamma_3 \leftrightarrow \gamma_4$  and  $\gamma_{35} \leftrightarrow \gamma_{45}$ .

$P_L + P_R = 1$ , we may decompose the four-component spinor  $\psi$  into left- and right-handed Weyl spinors  $\psi_{L/R} := P_{L/R}\psi$  and  $\bar{\psi}_{L/R} := \bar{\psi}P_{L/R}$ , each now representing two fermionic degrees of freedom. [On the honeycomb lattice  $\psi_L$  ( $\psi_R$ ) represents the quasiparticle excitations near the left (right) Dirac cone.] The Gross-Neveu action then decomposes into the two independent parts

$$S_{GN} = \int d^d x \left[ \bar{\psi}_L^a i\gamma_\mu \partial_\mu \psi_L^a + \frac{\bar{g}}{2N_f} (\bar{\psi}_L^a \psi_L^a)^2 \right] + (L \leftrightarrow R), \quad (5)$$

without any mixing between left- and right-handed spinors. (Note that, in contrast to the usual definition of the Weyl spinors in four dimensions, the chiral projector  $P_{L/R}$  commutes with  $\gamma_0$ , such that  $\bar{\psi}_L \psi_R \equiv 0$ .) Eq. (5) is evidently invariant under flavor rotations of the Weyl spinors:

$$\psi_L^a \mapsto U_L^{ab} \psi_L^b, \quad \bar{\psi}_L^a \mapsto \bar{\psi}_L^b (U_L^\dagger)^{ba}, \quad (6)$$

$$\psi_R^a \mapsto U_R^{ab} \psi_R^b, \quad \bar{\psi}_R^a \mapsto \bar{\psi}_R^b (U_R^\dagger)^{ba}, \quad (7)$$

where the two unitary matrices  $U_{L/R} \in U(N_f)$  may be chosen independently. This constitutes the Weyl representation of the  $U(N_f) \times U(N_f)$  symmetry. By separating the trace and the traceless part of the symmetry generator, we may as well split the symmetry group as

$$U(N_f) \times U(N_f) \simeq SU(N_f) \times SU(N_f) \times U(1)_V \times U(1)_A, \quad (8)$$

where the phase rotations  $U(1)_V$  (axial transformations  $U(1)_A$ ) correspond to the transformation (6)–(7) with scalar matrices  $U_{L/R} = e^{i\theta} \mathbb{1}$  ( $U_{L/R} = e^{\pm i\theta} \mathbb{1}$ ); the  $SU(N_f)$  factors are the remaining transformations with traceless generators. For the graphene system with  $N_f = 2$ , these factors have the following physical meaning:  $U(1)_V$  corresponds to charge conservation,  $U(1)_A$  denotes the translational symmetry on the honeycomb lattice [22], and the two  $SU(2)$  factors correspond to independent spin rotations in the two Dirac-cone sectors [23]. While the two  $U(1)$  symmetries are expected to hold in an effective low-energy theory of the honeycomb lattice system, the latter may possibly (for large coupling) not. Microscopically, only the single  $SU(2)$  transformation rotating the spin simultaneously in both sectors is a symmetry. The second  $SU(2)$  factor is, however, believed to be emergent if the on-site interaction does not become too large [21, 23].

*$\mathbb{Z}_2$  chiral symmetry:*

$$\psi \mapsto \gamma_5 \psi, \quad \bar{\psi} \mapsto -\bar{\psi} \gamma_5. \quad (9)$$

The analogous discrete chiral symmetry with  $\gamma_3$  instead of  $\gamma_5$  can be obtained by combining (9) with a prior chiral transformation (4) with fixed  $\theta = \pi/2$  [51].

*Parity symmetry:* We define parity transformation by inverting *one* spatial coordinate,  $x = (x_0, x_1, x_2) \mapsto x' = (x_0, -x_1, x_2)$ ,

$$\psi(x) \mapsto i\gamma_1 \gamma_5 \psi(x'), \quad \bar{\psi}(x) \mapsto \bar{\psi}(x') i\gamma_1 \gamma_5. \quad (10)$$

Note that different definitions of parity symmetry are in principle possible, when combining (10) with chiral transformations (4). On the honeycomb lattice, the above form corresponds to the reflection symmetry which exchanges the two Dirac points while not exchanging the sublattice labels [22]. The second reflection symmetry of the honeycomb lattice, which exchanges the spinor components belonging to the two different sublattices, is obtained by a combination of (9), (10), and the rotational symmetry from (2) with  $\epsilon_{\mu\nu} = \pi(\delta_{\mu 1} \delta_{\nu 2} - \delta_{\mu 2} \delta_{\nu 1})$ .

Further discrete transformations may be defined, such as time reversal and charge conjugation, both of which leave the Gross-Neveu action (1) invariant. For simplicity, we do not list them here, since they do not lead to any further restraints on possible operators in the theory. The  $U(N_f) \times U(N_f)$  Gross-Neveu theory space is uniquely determined by the above given set of symmetry transformations.

We now classify all pointlike operators up to the four-fermion level with respect to their symmetry. Let us start with the two-fermion terms, which represent the building blocks of the higher-order operators. Flavor symmetry ensures the form

$$\bar{\psi}^a \mathcal{O} \psi^a \quad (11)$$

with a  $4 \times 4$  matrix  $\mathcal{O}$ . A basis in the 16-dimensional space of  $4 \times 4$  operators is given by the gamma matrices and their products:

$$\mathcal{O} \in \text{span}\{\mathbb{1}_4, \gamma_\mu, \gamma_3, \gamma_5, \gamma_{\mu\nu}, \gamma_{35}, i\gamma_\mu \gamma_3, i\gamma_\mu \gamma_5\} \quad (12)$$

The two-fermion term (11) will be invariant under the continuous chiral symmetry (4), if  $[\mathcal{O}, \gamma_{35}] = 0$ , which restricts  $\mathcal{O} \in \text{span}\{\mathbb{1}_4, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{35}\}$ . For invariance under discrete chiral symmetry (9), however, we demand  $\mathcal{O}$  to anticommute with  $\gamma_5$ , which requires  $\mathcal{O} \in \text{span}\{\gamma_\mu, \gamma_3, \gamma_{35}, i\gamma_\mu \gamma_5\}$ . Finally, parity invariance implies the commutation relation  $[\mathcal{O}, i\gamma_1 \gamma_5] = 0$  and thus  $\mathcal{O} \in \text{span}\{\mathbb{1}_4, \gamma_0, \gamma_2, \gamma_3, \gamma_{02}, i\gamma_0 \gamma_3, i\gamma_2 \gamma_3, i\gamma_1 \gamma_5\}$ . If we additionally require Lorentz invariance, these restrictions mutually exclude each other, and there exists no two-fermion term that is invariant under all symmetries of the  $U(N_f) \times U(N_f)$  Gross-Neveu model. If we relax the condition of Lorentz symmetry down to the invariance under just spatial rotations, the only invariant two-fermion term would be  $\bar{\psi} \gamma_0 \psi$ . In the honeycomb-lattice system this represents the quasiparticle density, which vanishes at half filling.

On the level of four-fermion terms there are in principle two different types possible,

$$(\bar{\psi}^a \mathcal{O} \psi^a)(\bar{\psi}^b \mathcal{Q} \psi^b) \quad \text{and} \quad (\bar{\psi}^a \mathcal{O} \psi^b)(\bar{\psi}^b \mathcal{Q} \psi^a), \quad (13)$$

which we will refer to as having singlet and nonsinglet flavor structure, respectively. Not all of these terms, however, are independent: With the help of Fierz identities, we can always rewrite terms of the second type (with nonsinglet flavor structure) as a linear combination of terms of the first type (with singlet flavor struc-



ture), see below. To begin with, it thus suffices to determine the invariant terms with singlet flavor structure; the nonsinglet-type ones can subsequently be obtained by Fierz identities. From the above discussion it is clear, that only the terms with  $\mathcal{O} = \mathcal{Q}$  will have a chance to be invariant under the given symmetries: Each of the (anti)commutation conditions, which lead to restrictions of the two-fermion terms,  $[\mathcal{O}, \gamma_{35}] = 0$  (continuous chiral symmetry),  $\{\mathcal{O}, \gamma_5\} = 0$  (discrete chiral symmetry), and  $[\mathcal{O}, \gamma_{15}]$  (parity symmetry), divides the 16-dimensional space of  $4 \times 4$  matrices into two equally large 8 dimensional subspaces of matrices which do and do not respectively fulfill the particular (anti)commutation relation. Invariance of the four-fermion term (13) requires  $\mathcal{O}$  and  $\mathcal{Q}$  to be in the same subspace for each of the symmetries. Together with Lorentz invariance, this is not simultaneously achievable for all three above symmetries if  $\mathcal{O} \neq \mathcal{Q}$ . Furthermore, for the flavor singlet term in Eq. (13) to be invariant under the continuous chiral symmetry we need  $\mathcal{O} = \mathcal{Q}$  to commute with  $\gamma_{35}$ . A basis of flavor-singlet four-fermion terms invariant under the above set of symmetries is therefore

$$(S)^2 := (\bar{\psi}^a \psi^a)^2, \quad (P)^2 := (\bar{\psi}^a \gamma_{35} \psi^a)^2, \quad (14)$$

$$(V)^2 := (\bar{\psi}^a \gamma_\mu \psi^a)^2, \quad (T)^2 := \frac{1}{2} (\bar{\psi}^a \gamma_{\mu\nu} \psi^a)^2. \quad (15)$$

Our theory space includes several previously investigated systems:

- (1) The Gross-Neveu model in the four-dimensional reducible representation of the Clifford algebra (“reducible Gross-Neveu model”) with Lagrangian  $\mathcal{L} = \bar{\psi} i \gamma_\mu \partial_\mu \psi + \bar{g}_S (S)^2$  has been discussed in Ref. [12].
- (2) If we choose a gamma-matrix basis in which  $\gamma_{35} = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix}$  the  $N_f$  four-component Dirac spinors  $\psi^a$ ,  $a = 1, \dots, N_f$ , can be reduced to  $2N_f$  two-component Dirac spinors  $\chi^i$ ,  $i = 1, \dots, 2N_f$  by means of

$$\psi \equiv \begin{pmatrix} \chi^a \\ \chi^{a+N_f} \end{pmatrix} \quad \text{and} \quad \bar{\psi} \equiv \begin{pmatrix} \bar{\chi}^a \\ -\bar{\chi}^{a+N_f} \end{pmatrix}. \quad (16)$$

$\chi$  and  $\bar{\chi}$  are related to the Weyl spinors by  $\psi_L^a \pm \psi_R^a = \begin{pmatrix} \chi^a \\ \pm \chi^{a+N_f} \end{pmatrix}$  and  $\bar{\psi}_L^a \pm \bar{\psi}_R^a = \begin{pmatrix} \bar{\chi}^a \\ \mp \bar{\chi}^{a+N_f} \end{pmatrix}$ . In terms of the  $2N_f$  flavors of two-component spinors  $\chi^i$  the interaction channel  $(P)^2$  can be rewritten as a (standard) Gross-Neveu interaction [3],

$$(P)^2 = (\bar{\chi}^i \chi^i)^2. \quad (17)$$

The Lagrangian  $\mathcal{L} = \bar{\psi} i \gamma_\mu \partial_\mu \psi + \bar{g}_P (P)^2$  thus describes the Gross-Neveu model in the irreducible representation of the Clifford algebra (“irreducible Gross-Neveu model”). Its critical behavior has been studied in Refs. [5–10].

- (3) The system with Lagrangian  $\mathcal{L} = \bar{\psi} i \gamma_\mu \partial_\mu \psi + \bar{g}_V (V)^2$  is known as the Thirring model and has also been subject of several previous investigations [3, 14–20].

An analogous discussion for the flavor nonsinglet terms [second term in Eq. (13)] is possible, but can be kept short with the help of the Fierz identities [15]:

$$(S)^2 = -\frac{1}{4} [(S^D)^2 + (P^D)^2 + (V^D)^2 + (A^D)^2], \quad (18)$$

$$(P)^2 = -\frac{1}{4} [(S^D)^2 - (P^D)^2 + (V^D)^2 - (A^D)^2], \quad (19)$$

$$(V)^2 = -\frac{1}{4} [3(S^D)^2 - 3(P^D)^2 - (V^D)^2 + (A^D)^2], \quad (20)$$

$$(T)^2 = -\frac{1}{4} [3(S^D)^2 + 3(P^D)^2 - (V^D)^2 - (A^D)^2], \quad (21)$$

where we have abbreviated

$$(S^D)^2 := (\psi^a \psi^b)^2 + (\psi^a \gamma_{35} \psi^b)^2, \quad (22)$$

$$(P^D)^2 := (\psi^a \gamma_3 \psi^b)^2 + (\psi^a \gamma_5 \psi^b)^2, \quad (23)$$

$$(V^D)^2 := (\psi^a \gamma_\mu \psi^b)^2 + \frac{1}{2} (\psi^a \gamma_{\mu\nu} \psi^b)^2, \quad (24)$$

$$(A^D)^2 := (\psi^a i \gamma_\mu \gamma_3 \psi^b)^2 + (\psi^a i \gamma_\mu \gamma_5 \psi^b)^2. \quad (25)$$

This yields an (invertible) one-to-one correspondence between vectors in the space of flavor singlet terms with their “dual” counterparts in the space of flavor nonsinglet terms. Therefore, any invariant flavor nonsinglet term must be a linear combination of the dual basis vectors  $(S^D)^2$ ,  $(P^D)^2$ ,  $(V^D)^2$ , and  $(A^D)^2$ . A full basis of fermionic four-point functions in the limit of point-like interactions is thus given by, e.g., the four terms in Eqs. (14)–(15) or the four terms in Eqs. (22)–(25), or a combination thereof. Put differently, any four-fermion theory with the symmetries of the  $U(N_f) \times U(N_f)$  Gross-Neveu model represents one point in the theory space spanned by a set of four basis vectors from Eqs. (14)–(15), (22)–(25). In the following, we will investigate the RG evolution in this theory space with a particular focus on possible fixed points.

Before proceeding, however, let us make a comment on the special case of single fermion flavor ( $N_f = 1$ ), which in the condensed-matter applications corresponds to the common simplified model of spinless electrons. In this case, we have two further relations between the interaction terms

$$(S^D)^2 = (\bar{\psi} \psi)^2 + (\bar{\psi} \gamma_{35} \psi)^2 = (S)^2 + (P)^2, \quad (26)$$

$$(V^D)^2 = (\bar{\psi} \gamma_\mu \psi)^2 + \frac{1}{2} (\bar{\psi} \gamma_{\mu\nu} \psi)^2 = (V)^2 + (T)^2, \quad (27)$$

in addition to the Fierz identities. Evaluation of the corresponding matrix rank yet shows that only one of these two is independent and, e.g., Eq. (26) can be obtained by linear combination of Eq. (27) with the Fierz identities (18)–(21). Thus, for  $N_f = 1$  a “Fierz-complete” basis is given by just three independent interaction terms, e.g., the flavor singlets  $(S)^2$ ,  $(P)^2$ , and  $(V)^2$ . The corresponding relation among the singlet invariants is

$$(T)^2 = -3(S)^2 - 3(P)^2 - (V)^2, \quad \text{for } N_f = 1. \quad (28)$$

This is in agreement with the previous study of spinless fermions on the honeycomb lattice [22].

### III. FUNCTIONAL RENORMALIZATION GROUP

In order to compute the RG beta functions, we use the functional renormalization group in terms of Wetterich's evolution equation for the effective average action  $\Gamma_k$  [39],

$$\partial_k \Gamma_k = \frac{1}{2} \text{STr} \left[ \partial_k R_k (\Gamma_k^{(2)} + R_k)^{-1} \right], \quad (29)$$

where  $\Gamma_k$  denotes a scale-dependent effective action as a function of an infrared RG scale  $k \in [0, \Lambda]$  ( $\Lambda$  being the UV cutoff). This action interpolates between the microscopic action  $S$  for  $k \rightarrow \Lambda$ , and the full quantum effective action  $\Gamma$  (generating functional of one-particle irreducible Green's functions) for  $k \rightarrow 0$ . The quantity  $\Gamma_k^{(2)}$  denotes the corresponding Hessian of the effective action, and the function regulator  $R_k$  defines the details of the regularization procedure. Within the approximation used in this work, all our results given below turn out to be independent of the regularization scheme. For reviews on the functional RG, see Refs. [40, 52, 53].

While the Wetterich equation (29) is an exact identity for the effective average action, it is difficult to solve it without the use of suitable approximation schemes. The four-fermion theories considered in this work exhibit a lower critical dimension of  $d = 2$ , at which the four-fermion couplings become marginal. In dimension  $d = 2 + \epsilon$  for small  $\epsilon$  any interacting fixed point will therefore be of the order  $g_* = \mathcal{O}(\epsilon)$  and thus accessible via a (renormalized) perturbative expansion. An analogous argument can be made for a large number of fermion flavors  $N_f$ . Any higher-order fermionic term, as well as momentum-dependent fermionic vertices will then be irrelevant at such an interacting fixed point, and the effective average action in the vicinity of the fixed point will have the truncated form

$$\Gamma_k = \int d^d x \left[ \bar{\psi}^a i \gamma_\mu \partial_\mu \psi^a + \frac{\bar{g}_{S,k}}{2N_f} (S)^2 + \frac{\bar{g}_{P,k}}{2N_f} (P)^2 + \frac{\bar{g}_{V,k}}{2N_f} (V)^2 + \frac{\bar{g}_{T,k}}{2N_f} (T)^2 \right], \quad (30)$$

where we have used the flavor-singlet basis from Eqs. (14)–(15) with scale-dependent four-fermion couplings  $\bar{g}_{S,k}$ ,  $\bar{g}_{P,k}$ ,  $\bar{g}_{V,k}$ , and  $\bar{g}_{T,k}$ . For  $N_f = 1$ , the last term  $\propto (T)^2$  in Eq. (30) can be rewritten as a linear combination of the three former terms  $(S)^2$ ,  $(P)^2$ , and  $(V)^2$ , and  $\Gamma_k$  would be spanned by just three couplings. We have neglected the possibility of a wave-function renormalization for the fermions, as the fermion anomalous dimension is known to vanish in the limit of pointlike fermionic interactions [15, 48]. In the following, we will use Eq. (30) as an ansatz for  $\Gamma_k$  to solve the Wetterich equation (29) approximately. As all omitted terms are perturbatively RG irrelevant, this truncation will become exact to first order in  $\epsilon$  close to the lower critical dimension, as well as for large number of flavors  $N_f$ . For the physical cases with  $\epsilon = 1$  and small  $N_f$ , Eq. (30) can of course only be

viewed as the simplest possible truncation within, e.g., a systematic expansion of  $\Gamma_k$  in terms of derivatives. Beyond the perturbative domain, higher-order terms (e.g., momentum-dependent terms) would in principle have to be taken into account, and the stability of our results against the inclusion of such terms would have to be verified. In the framework of the functional RG this can be done, e.g., by means of partial [23, 42] or dynamical [3, 43, 44, 54] bosonization techniques, or by various decomposition schemes in momentum space [40, 41]. A third option is offered by working with full potentials for fermion bilinears and determining the solution of the flow on all scales on a larger function space including *weak solutions* [55], as has been used for 3d models in [48]. This will be left for future work, and we will here confine ourselves to the study of the RG flow in the limit of pointlike (momentum-independent) four-fermion couplings.

### IV. GENERAL FORMULA FOR 4-FERMI BETA FUNCTIONS

By plugging the ansatz for  $\Gamma_k$  [Eq. (30)] into the Wetterich equation (29), the flow equations for the four-fermion couplings can straightforwardly (though possibly somewhat tediously) be obtained by equating the coefficients on the left and right hand side of the equation. However, it appears worthwhile to cast Eq. (30) into a more general form

$$\Gamma_k = \int d^d x \left[ \bar{\psi}^a i \gamma_\mu \partial_\mu \psi^a + \sum_i \frac{\bar{g}_{i,k}}{2N_f} (\bar{\psi}^a \mathcal{O}_i \psi^a)^2 \right], \quad (31)$$

describing a general massless fermion system interacting via several short-range interactions. Let us assume that the interactions in Eq. (31) represent a “full basis” of fermionic four-point functions in the pointlike limit in the sense of Sec. II, thus including all flavor-singlet terms which are invariant under a given symmetry. Invariant flavor-nonsinglet terms can be rewritten as a linear combination of the above by means of Fierz identities, and in that sense the basis is “Fierz complete”. As we shall see in this section, the functional RG equation will allow us to compute the beta functions for this general ansatz of four-fermion interactions. The resulting formula is readily automatable within a computer algebra system. This will allow to compute the one-loop beta functions in various fermion systems in arbitrary dimension. E.g., it should be applicable to derive effective theories for several condensed-matter systems featuring chiral fermions, such as single- [22] or bilayer [45] graphene, or in the (3+1)-dimensional systems with linear [26] or quadratic [46] dispersion relation—systems in which generically a large number of interactions are compatible with the given microscopic symmetries.<sup>2</sup>

<sup>2</sup> We note that while we here have assumed relativistic fermions, the crucial point for the derivation of the beta functions is the

In order to formalize the procedure of equating coefficients, we introduce the “projector”

$$\hat{P}_i := \sum_j \sum_{a,b} \sum_{\alpha,\beta,\gamma,\delta} c_{ij} \mathcal{O}_{j,\gamma\beta} \mathcal{O}_{j,\delta\alpha} \times \frac{\overrightarrow{\delta}}{\delta\psi_\alpha^a} \frac{\overrightarrow{\delta}}{\delta\psi_\beta^b} \left( \cdot \right) \frac{\overleftarrow{\delta}}{\delta\psi_\gamma^b} \frac{\overleftarrow{\delta}}{\delta\psi_\delta^a} \Big|_{\bar{\psi}=\psi=0} \quad (32)$$

with to-be-defined coefficients  $c_{ij}$ . In the functional derivatives, the Greek letters  $\alpha, \beta, \gamma, \delta = 1, \dots, d_\gamma$  refer to the spinor index, while Latin letters  $a, b = 1, \dots, N_f$  denote the flavor index.  $d_\gamma$  is the dimension of the representation of the Clifford algebra,  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \mathbb{1}_{d_\gamma}$ . When applied to our ansatz for the effective action [Eq. (31)],  $\hat{P}_i$  gives

$$\hat{P}_i \Gamma = \sum_{j,l} c_{ij} M_{jl} \bar{g}_l \quad (33)$$

with the symmetric matrix

$$M_{jl} = N_f [\text{Tr}(\mathcal{O}_j \mathcal{O}_l)]^2 - \text{Tr}(\mathcal{O}_j \mathcal{O}_l \mathcal{O}_j \mathcal{O}_l) \quad (\text{no sum}). \quad (34)$$

(Wherever possible, we suppress the scale index  $k$  of the coupling  $\bar{g}_l \equiv \bar{g}_{l,k}$  in the following to avoid confusion.) If we choose the coefficients  $c_{ij}$  such that

$$\sum_j c_{ij} M_{jl} = \delta_{il} \quad \Rightarrow \quad c_{ij} = (M^{-1})_{ij}, \quad (35)$$

we find that by applying  $\hat{P}_i$  on the Wetterich equation we “extract” the beta function for the  $i$ th coupling constant,

$$\begin{aligned} \partial_k \bar{g}_i &= \hat{P}_i (\partial_k \Gamma_k) \\ &= \frac{1}{2} \hat{P}_i \text{STr} \left[ \partial_k R_k (\Gamma_k^{(2)} + R_k)^{-1} \right]. \end{aligned} \quad (36)$$

This can be further simplified by introducing the scale-derivative  $\tilde{\partial}_k := (\partial_k R_k) \frac{\delta}{\delta R_k}$ , which acts only on the regulator’s  $k$ -dependence [52],

$$\partial_k R_k (\Gamma_k^{(2)} + R_k)^{-1} = \tilde{\partial}_k \ln(\Gamma_k^{(2)} + R_k), \quad (37)$$

and expanding the logarithm in powers of interactions

$$\begin{aligned} \ln(\Gamma_k^{(2)} + R_k) &= \ln(\Gamma_{k,0}^{(2)} + R_k) \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[ \Delta \Gamma_k^{(2)} (\Gamma_{k,0} + R_k)^{-1} \right]^n. \end{aligned} \quad (38)$$

Here we have split the fluctuation matrix into its field-independent propagator part  $\Gamma_{k,0}^{(2)} = \Gamma_k^{(2)}|_{\bar{\psi}=\psi=0}$  (which is straightforwardly invertible) and the fluctuation part  $\Delta \Gamma_k^{(2)} = \Gamma_k^{(2)} - \Gamma_{k,0}^{(2)}$ . For the four-fermion theories,  $\Delta \Gamma_k^{(2)}$  is quadratic in  $\psi, \bar{\psi}$ , and thus the series in Eq. (38) terminates after  $n = 2$  when applying the “projector”  $\hat{P}_i$ . In massless systems also the leading term for  $n = 1$  vanishes for reasons of symmetry. The only nonvanishing term when plugging Eqs. (37) and (38) into (36) is then the quadratic term,

$$\partial_k \bar{g}_i = -\frac{1}{4} \hat{P}_i \tilde{\partial}_k \text{STr} \left[ \Delta \Gamma_k^{(2)} (\Gamma_{k,0} + R_k)^{-1} \right]^2, \quad (39)$$

ensuring that the flow of the four-fermi coupling has the usual (one-loop) quadratic form

$$\partial_k \bar{g}_i = \frac{4v_d}{N_f k^{d-1}} \ell_1^{(F)d} \sum_{j,l} \bar{g}_j A_{i,jl} \bar{g}_l, \quad (40)$$

with the coefficient matrix  $A_{i,jl}$  to be determined. Here, we have used the standard abbreviations  $v_d = \frac{1}{4} \text{vol}(S^{d-1}) / (2\pi)^d = [2^{d+1} \pi^{d/2} \Gamma(d/2)]^{-1}$ , which arises from the angular part of the loop integral, and the dimensionless threshold function  $\ell_1^{(F)d}$ , representing the loop integral’s radial part. It incorporates the regulator dependence and can be defined by [15]

$$\ell_1^{(F)d} = -\partial_k \int_0^{\Lambda^2} dq^2 \frac{q^{d-4}}{[1 + r(q^2/k^2)]^2}. \quad (41)$$

In terms of rescaled dimensionless coupling constants  $g_i := 4v_d \ell_1^{(F)d} k^{d-2} \bar{g}_i$  the beta function reads

$$\partial_t g_i = (d-2)g_i + \frac{1}{N_f} \sum_{j,l} g_j A_{i,jl} g_l, \quad t = \ln(k/\Lambda). \quad (42)$$

By employing the ansatz (31) for  $\Gamma_k$ , Eq. (39) gives after some elementary algebra the coefficient matrix

$$\begin{aligned} A_{i,jl} &= \frac{1}{d} \sum_m c_{im} \left\{ \text{Tr}(\mathcal{O}_l \gamma_\mu \mathcal{O}_j \mathcal{O}_m \mathcal{O}_l \gamma_\mu \mathcal{O}_j \mathcal{O}_m) \right. \\ &\quad - \text{Tr}(\mathcal{O}_l \gamma_\mu \mathcal{O}_j \mathcal{O}_m \mathcal{O}_j \gamma_\mu \mathcal{O}_l \mathcal{O}_m) \\ &\quad - \text{Tr}(\mathcal{O}_j \mathcal{O}_m \mathcal{O}_l \gamma_\mu \mathcal{O}_j \gamma_\mu \mathcal{O}_l \mathcal{O}_m) \\ &\quad - \text{Tr}(\mathcal{O}_l \mathcal{O}_m \mathcal{O}_j \gamma_\mu \mathcal{O}_l \gamma_\mu \mathcal{O}_j \mathcal{O}_m) \\ &\quad + N_f [-\text{Tr}(\mathcal{O}_l \gamma_\mu \mathcal{O}_j \mathcal{O}_m) \text{Tr}(\mathcal{O}_l \gamma_\mu \mathcal{O}_j \mathcal{O}_m) \\ &\quad + \text{Tr}(\mathcal{O}_l \gamma_\mu \mathcal{O}_j \mathcal{O}_m) \text{Tr}(\mathcal{O}_j \gamma_\mu \mathcal{O}_l \mathcal{O}_m) \\ &\quad + \text{Tr}(\mathcal{O}_j \mathcal{O}_m \mathcal{O}_l \mathcal{O}_m) \text{Tr}(\gamma_\mu \mathcal{O}_j \gamma_\mu \mathcal{O}_l) \\ &\quad + \text{Tr}(\mathcal{O}_j \mathcal{O}_m) \text{Tr}(\mathcal{O}_l \gamma_\mu \mathcal{O}_j \gamma_\mu \mathcal{O}_l \mathcal{O}_m) \\ &\quad + \text{Tr}(\mathcal{O}_l \mathcal{O}_m) \text{Tr}(\mathcal{O}_j \gamma_\mu \mathcal{O}_l \gamma_\mu \mathcal{O}_j \mathcal{O}_m)] \\ &\quad \left. - N_f^2 \text{Tr}(\mathcal{O}_j \mathcal{O}_m) \text{Tr}(\mathcal{O}_l \mathcal{O}_m) \text{Tr}(\gamma_\mu \mathcal{O}_j \gamma_\mu \mathcal{O}_l) \right\}. \end{aligned} \quad (43)$$

Eqs. (42)–(43) represent the main result of this section. They are (up to the rescaling) exactly the one-loop

---

fermion propagator’s Dirac-matrix structure  $\sim \gamma_\mu$ , which has an analogous form in nonrelativistic chiral fermion systems with quadratic dispersion relation both in 2+1 [56] as well as 3+1 dimensions [46]. We believe that an analogous formula as derived in this section should also be possible to derive for these systems.

beta functions as they would have been obtained within the standard Wilsonian momentum-shell RG approach. In particular due to its simple possible implementation within a computer algebra system, we expect this general formula to be valuable also beyond the scope of the models considered in this work. We will use it in the following to compute the flow of the four couplings present in the  $U(N_f) \times U(N_f)$  Gross-Neveu theory space [Eq. (30)].

## V. FLOW EQUATIONS

A straightforward evaluation of the traces in Eq. (43) gives the beta functions for the pointlike four-fermion couplings in the four-dimensional Gross-Neveu theory space for  $N_f > 1$

$$\partial_t g_S = (d-2)g_S + \frac{1}{N_f} [(-4N_f+2)g_S^2 + g_S(2g_P+6g_V+6g_T)+8g_Vg_T], \quad (44)$$

$$\partial_t g_P = (d-2)g_P + \frac{1}{N_f} [(-4N_f+2)g_P^2 + g_P(2g_S+6g_V+6g_T)+4g_V^2+4g_T^2], \quad (45)$$

$$\partial_t g_V = (d-2)g_V + \frac{1}{N_f} \left[ \frac{4N_f+2}{3}g_V^2 + g_V \left( -\frac{2}{3}g_S+2g_P+\frac{2}{3}g_T \right) + \frac{8}{3}g_Sg_T \right], \quad (46)$$

$$\partial_t g_T = (d-2)g_T + \frac{1}{N_f} \left[ \frac{4N_f+2}{3}g_T^2 + g_T \left( -\frac{2}{3}g_S+2g_P+\frac{2}{3}g_V \right) + \frac{8}{3}g_Sg_V \right]. \quad (47)$$

We observe that these flow equations are symmetric under the exchange of  $g_V \leftrightarrow g_T$ . They generalize several previous RG approaches to relativistic fermion systems:

Setting  $g_P = g_V = g_T \equiv 0$  defines the  $U(N_f) \times U(N_f)$  Gross-Neveu model with four-component Dirac fermions. The remaining flow equation

$$\partial_t g_S = (d-2)g_S - \frac{4N_f-2}{N_f}g_S^2 \quad (48)$$

agrees with the previous calculation [12]. We also see that this subspace is closed under the RG, and  $g_P$ ,  $g_V$ , and  $g_T$  will not be generated if they all vanish at the initial scale.

The subspace  $g_S = g_T \equiv 0$  has also been considered earlier, and our flow equations agree with the former work [15]. In this subspace the system's  $\mathbb{Z}_2 \times U(N_f) \times U(N_f)$  chiral symmetry is elevated to  $U(2N_f)$ , generated by the  $4N_f^2$  matrices  $\{\lambda_0, \dots, \lambda_{N_f^2-1}\} \otimes \{\mathbb{1}_4, \gamma_3, \gamma_5, \gamma_{35}\}$ , and RG closedness is protected by symmetry. This is the theory space of the Thirring model in  $2 < d < 4$  dimensions [3, 15]. In the following, we will refer to it as  $U(2N_f)$  or Thirring subspace.

For  $N_f = 1$ , when the theory space is just three-dimensional, one of the four couplings in Eqs. (44)–(47)

can be completely eliminated. Using Eq. (28), we find

$$g_S(S)^2 + g_P(P)^2 + g_V(V)^2 + g_T(T)^2 = (g_S - 3g_T)(S)^2 + (g_P - 3g_T)(P)^2 + (g_V - g_T)(V)^2. \quad (49)$$

If we shift the couplings as

$$g_S - 3g_T \mapsto g_S, \quad g_P - 3g_T \mapsto g_P, \quad g_V - g_T \mapsto g_V, \quad (50)$$

the corresponding shifted beta functions for  $N_f = 1$  then indeed become independent of the fourth coupling  $g_T$ ,

$$\partial_t g_S = (d-2)g_S + 2g_S(-g_S + g_P - g_V), \quad (51)$$

$$\partial_t g_P = (d-2)g_P + 2g_P(-g_P + g_S + 3g_V) + 4g_V(g_V - 2g_S), \quad (52)$$

$$\partial_t g_V = (d-2)g_V + 2g_V(g_V - \frac{5}{3}g_S + g_P). \quad (53)$$

We note that Eqs. (51)–(53) are equivalent to the previous one-loop flow equations for the spinless-fermion system on the honeycomb lattice [22].<sup>3</sup> The flow equations thus again reflect the fact that for  $N_f = 1$  different values for the coupling  $g_T$  (in terms of the shifted couplings) do not correspond to different physical points in theory space, but rather can be mapped onto each other by means of the Fierz identities; the linearly dependent set of the four elements  $(S)^2$ ,  $(P)^2$ ,  $(V)^2$ , and  $(T)^2$ , which span the theory space, is no longer a basis for  $N_f = 1$ , but an overcomplete *frame*, as used, e.g., in the theory of signal processing [57]. Of course, the same would happen for  $N_f > 1$  if further linearly dependent (e.g., flavor-nonsinglet) terms were to be added to the effective action.

Let us express a warning at this point: When counting the number of (physically distinct) fixed points, and the number of their accompanying relevant and irrelevant directions (as we shall do in the following) it is thus important to take all identities between the fermionic terms into account and reduce the frame to a (linearly independent) *basis*: Otherwise, e.g., critical fixed points, having a single relevant direction in the irreducible basis, may develop spurious additional relevant (or irrelevant) directions in the redundant directions, preventing them to be identified as critical fixed points in the overcomplete frame. On the other hand, if by means of a single- or few-channel approximation we *a priori* neglect particular (physical) directions in theory space, which in principle would be compatible with the symmetry, a fixed point with several RG relevant directions could falsely appear as critical fixed point in such a truncated description. In this work, we therefore advocate the use of a Fierz-complete irreducible basis of theory space in order to find the number of fixed points and their relevant and irrelevant directions.

<sup>3</sup> The equivalence can be readily seen by using the coupling transformation  $g_\alpha = -g_V$ ,  $g_{C1} = g_S - g_V$ , and  $g_{D2} = g_P - 2g_V$ , where  $g_{\alpha, C1, D2}$  are the three couplings used in Ref. [22], assuming the full Lorentz symmetry.





TABLE I. Fixed points for  $N_f = 1$ , their locations, and number of relevant directions. GN: Gross-Neveu.

	$(g_S^*, g_V^*, g_P^*)$	$\#(\Theta > 0)$	
$\mathcal{O}$	$(0, 0, 0)$	0	Gaussian
$\mathcal{A}$	$(0, 0, 1/2)$	1	irreducible GN
$\mathcal{B}$	$(0, -\frac{3+\sqrt{5}}{4}, \frac{\sqrt{5}+1}{4})$	2	
$\mathcal{C}$	$(0, -\frac{3-\sqrt{5}}{4}, -\frac{\sqrt{5}-1}{4})$	1	Thirring
$\mathcal{D}$	$(1/2, 0, 0)$	2	reducible GN
$\mathcal{E}$	$(\frac{3(3-\sqrt{5})}{4}, \frac{3-\sqrt{5}}{4}, \frac{5-2\sqrt{5}}{2})$	1	
$\mathcal{F}$	$(\frac{3(3+\sqrt{5})}{4}, \frac{3+\sqrt{5}}{4}, \frac{5+2\sqrt{5}}{2})$	2	
$\mathcal{G}$	$(\infty, 0, \infty)$	3	diverges for $N_f \rightarrow 1$

points  $\mathcal{O}$ ,  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  lie in the two-dimensional, and thus RG-invariant, subspace with  $g_S = 0$  (yellow/light gray plane in Fig. 1). This plane is in fact the Thirring subspace with  $U(2N_f)$  symmetry. The fixed-point structure of the  $U(2N_f)$  theory space has been investigated earlier [3, 15, 20], showing that  $\mathcal{A}$  and  $\mathcal{C}$  are critical fixed points with exactly one relevant direction *within this plane*.  $\mathcal{A}$  is the fixed point which describes the Gross-Neveu model in the irreducible two-dimensional representation of the Clifford algebra with a four-fermi interaction that can be rewritten in terms of two-component (Weyl) spinors  $\chi$  as  $(P)^2 = (\bar{\psi}\gamma_{35}\psi)^2 \propto (\bar{\chi}\chi)^2$  [Eq. (17)]. The critical behavior of  $\mathcal{A}$  is well-known [5–10]. The fixed point  $\mathcal{C}$  describes the chiral phase transition expected in the three-dimensional Thirring model and as such has been dubbed Thirring fixed point [3, 15, 20]—although we emphasize that  $\mathcal{C}$  does not lie on the Thirring axis with pure  $(V)^2 = (\bar{\psi}\gamma_\mu\psi)^2$  interaction, but its fixed-point action includes both  $(P)^2$  and  $(V)^2$  contributions [15]. The critical behavior of  $\mathcal{C}$  has recently also been approached [3, 20]. Here, we are able to determine whether or not small perturbations *out* of the  $U(2N_f)$ -symmetric plane are relevant in the sense of the RG. From Eqs. (51)–(53) we find the following critical exponents:

$$\mathcal{A}: \quad \Theta = (1, -2, -2), \quad (58)$$

$$\mathcal{C}: \quad \Theta = (1, -3 + \sqrt{5}, -3\sqrt{5} + 5), \quad (59)$$

and therefore both  $U(2N_f)$  critical fixed points remain critical also in the lower-symmetric  $U(N_f) \times U(N_f)$  theory space. In the vicinity of both  $\mathcal{A}$  at  $(g_S^*, g_V^*, g_P^*) = (0, 0, \frac{1}{2})$  and  $\mathcal{C}$  at  $(0, -\frac{3-\sqrt{5}}{4}, -\frac{\sqrt{5}-1}{4})$  any  $U(2N_f)$ -breaking but  $U(N_f) \times U(N_f)$ -preserving perturbation is thus RG irrelevant for  $N_f = 1$ , since the only relevant directions of each fixed point are the fixed-point vectors  $\mathbf{g}_\mathcal{A}^*$  and  $\mathbf{g}_\mathcal{C}^*$  itself.

A second RG-invariant plane is given by  $g_V = 0$  (cyan/dark gray plane in Fig. 1), and it contains besides  $\mathcal{A}$  the non-Gaussian fixed point  $\mathcal{D}$  at  $(g_S^*, g_V^*, g_P^*) = (\frac{1}{2}, 0, 0)$ .  $\mathcal{D}$  describes a theory with pure  $(\bar{\psi}\psi)^2$  four-fermi interaction, i.e., the Gross-Neveu model in the reducible, four-dimensional, representation of the Clifford-Algebra [12]. The 8th solution  $\mathcal{G}$  of the fixed-point equations that

diverges for  $N_f \rightarrow 1$  also lies in this subplane at  $(g, 0, g)$  with  $g \propto (N_f - 1)^{-1}$ . The line  $(g_S, g_V, g_P) = \lambda(1, 0, 1)$  is therefore also RG invariant, just as the Gross-Neveu axes  $\lambda(1, 0, 0)$  and  $\lambda(0, 0, 1)$ . As has been observed previously [22], both  $\mathcal{A}$  and  $\mathcal{D}$  have only one relevant direction *in this plane*. However, while perturbations out of this plane are irrelevant in the vicinity of  $\mathcal{A}$ , such is not the case in the vicinity of fixed point  $\mathcal{D}$ : For this fixed point the critical exponents are

$$\mathcal{D}: \quad \Theta = (1, 2/3, -2), \quad (60)$$

and thus the reducible-Gross-Neveu fixed point  $\mathcal{D}$  has *two* relevant directions in the  $U(N_f) \times U(N_f)$  theory space for  $N_f = 1$ . Since there is no higher symmetry that could forbid perturbations  $\propto g_V$  the fixed point  $\mathcal{D}$  is not a critical fixed point for  $N_f = 1$  and cannot describe a second-order phase transition, as long as only one microscopic parameter is tuned.

There is, however, a third critical fixed point, located at

$$\mathcal{E}: \quad (g_S^*, g_V^*, g_P^*) = \left( \frac{3(3-\sqrt{5})}{4}, \frac{3-\sqrt{5}}{4}, \frac{5-2\sqrt{5}}{2} \right), \quad (61)$$

which has the same critical exponents as fixed point  $\mathcal{C}$  in our approximation,

$$\mathcal{E}: \quad \Theta = (1, -3 + \sqrt{5}, -3\sqrt{5} + 5). \quad (62)$$

Small perturbations near fixed point  $\mathcal{D}$  drive the flow to either  $\mathcal{C}$  or  $\mathcal{E}$ , depending on the sign of  $g_V$ . We have summarized our results for the  $N_f = 1$  fixed-point structure in Tab. I.

## VIII. FIXED-POINT STRUCTURE FOR $N_f \geq 2$

### A. Collision of fixed points

For  $N_f \geq 2$  the additional flow equation renders the  $U(N_f) \times U(N_f)$  theory space four-dimensional, thus generating a more complex fixed-point structure. Generically, one expects  $2^4 = 16$  possibly complex or degenerate solutions of the fixed-point equations [Eqs. (44)–(47)]. We find that the number of real (and therefore physical) fixed points actually depends on the flavor number  $N_f$ . In the small- $N_f$  regime  $2 \leq N_f < N_f^{(1)}$  with

$$\begin{aligned} N_f^{(1)} &= -\frac{4}{9} + \frac{1}{9} \left( 9872 - 144\sqrt{3345} \right)^{\frac{1}{3}} \\ &\quad + \frac{2}{9} \left( 1234 + 18\sqrt{3345} \right)^{\frac{1}{3}} \\ &\simeq 3.76 \end{aligned}$$

there are 12 distinct real fixed points and two pairs of complex conjugate solutions. Above  $N_f \geq N_f^{(1)}$  both

TABLE II. Selected fixed points for  $N_f \geq 2$ , their locations, and number of relevant directions. At  $N_f \searrow N_f^{(1)} \simeq 3.76$  the three fixed points  $\mathcal{H}$ ,  $\mathcal{I}$ , and  $\mathcal{J}$  collide.  $\mathcal{I}$  and  $\mathcal{J}$  merge and subsequently become complex for  $N_f < N_f^{(1)}$ , and exchange stability of one direction with the fixed point  $\mathcal{H}$ . At  $N_f \nearrow N_f^{(2)} = 6$ ,  $\mathcal{I}$  collides with the Thirring fixed point  $\mathcal{C}$ , and  $\mathcal{J}$  collides with  $\mathcal{K}$ , again exchanging roles: for  $N_f > 6$ ,  $\mathcal{C}$  and  $\mathcal{K}$  become critical fixed points.  $\mathcal{A}$  and  $\mathcal{D}$  on the two (irreducible and reducible) Gross-Neveu (GN) axes  $\sim (\bar{\psi}\gamma_{35}\psi)^2$  and  $(\bar{\psi}\psi)^2$ , respectively, are critical fixed points for all  $N_f \geq 2$ .

	$(g_S^*, g_P^*, g_V^*, g_T^*)$	$\#(\Theta > 0)$	
$\mathcal{A}$	$(0, a, 0, 0)$	1	irreducible GN
$\mathcal{B}$	$(0, b_1, b_2, 0)$	$\begin{cases} 3, & N_f < 6 \\ 2, & N_f > 6 \end{cases}$	
$\mathcal{C}$	$(0, c_1, c_2, 0)$	$\begin{cases} 2, & N_f < 6 \\ 1, & N_f > 6 \end{cases}$	Thirring
$\mathcal{D}$	$(a, 0, 0, 0)$	1	reducible GN
$\mathcal{G}$	$(g, g, 0, 0)$	2	
$\mathcal{H}$	$(h_1, h_1, h_2, h_2)$	$\begin{cases} 1, & N_f < N_f^{(1)} \\ 2, & N_f > N_f^{(1)} \end{cases}$	
$\mathcal{I}$	$(i_1, i_2, i_3, i_4)$	$\begin{cases} 1, & N_f^{(1)} < N_f < 6 \\ 2, & N_f > 6 \end{cases}$	
$\mathcal{J}$	$(i_1, i_2, i_4, i_3)$	$\begin{cases} 1, & N_f^{(1)} < N_f < 6 \\ 2, & N_f > 6 \end{cases}$	
$\mathcal{K}$	$(0, c_1, 0, c_2)$	$\begin{cases} 2, & N_f < 6 \\ 1, & N_f > 6 \end{cases}$	
$\mathcal{L}$	$(l_1, l_1, l_2, l_2)$	4	

pairs simultaneously become real and we find 16 real fixed points. A selection of them is listed in Tab. II. For particular values of  $N_f$  these solutions become degenerate, i.e., fixed points approach each other in coupling space as a function of  $N_f$  and eventually collide. In general, when two fixed points collide, two principally different situations are possible: (1) The fixed points can “run through” each other as a function of  $N_f$  and exchange roles with respect to the RG stability of the axis connecting the two fixed points. This phenomenon is well known for the Wilson-Fisher fixed point in  $\phi^4$  theory as a function of space-time dimension  $d$ , which collides with the Gaussian fixed point for  $d \nearrow 4$  and becomes unstable for  $d > 4$  [1]. It is also observed for the multicritical fermionic fixed point on graphene’s honeycomb lattice as a function of flavor number  $N_f$  [59]. (2) The other possible situation is that the fixed points “merge” and eventually disappear into the complex plane. Such has been observed in many-flavor QCD [60], three-dimensional scalar [61] or fermionic [62] QED, and recently also in nonrelativistic systems with quadratic band touching [46].

In our system, we find two values of  $N_f$  for which a collision of fixed points occurs. The first one is particularly interesting from a general viewpoint: In the limit  $N_f \searrow N_f^{(1)}$  we in fact find two *triples* of solution, each consisting of *three* non-Gaussian fixed points that ap-

proach each other in coupling space and eventually collide simultaneously at two different locations. Let us discuss one of these triples in more detail. It consists of the three fixed points  $\mathcal{H}$ ,  $\mathcal{I}$ , and  $\mathcal{J}$ , which for general  $N_f > N_f^{(1)}$  are located at

$$\mathcal{H} : (g_S^*, g_P^*, g_V^*, g_T^*) = (h_1, h_1, h_2, h_2), \quad (63)$$

$$\mathcal{I} : (g_S^*, g_P^*, g_V^*, g_T^*) = (i_1, i_2, i_3, i_4), \quad (64)$$

$$\mathcal{J} : (g_S^*, g_P^*, g_V^*, g_T^*) = (i_1, i_2, i_4, i_3), \quad (65)$$

where  $h_{1,2} \equiv h_{1,2}(N_f)$ ,  $i_{1,\dots,4} \equiv i_{1,\dots,4}(N_f)$  ( $|i_3| \geq |i_4|$ ) are functions of  $N_f$ , which we do not display explicitly for reasons of readability. The symmetry between the locations of  $\mathcal{I}$  and  $\mathcal{J}$  is determined by the  $g_V \leftrightarrow g_T$  symmetry of the one-loop flow equations. For  $N_f \searrow N_f^{(1)}$  we find  $i_{1,2} \rightarrow h_1$  and  $i_{3,4} \rightarrow h_2$ , i.e., all three fixed points collide. Two of them ( $\mathcal{I}$  and  $\mathcal{J}$ ) merge and disappear into the complex plane for  $N_f < N_f^{(1)}$ , while the third ( $\mathcal{H}$ ) remains as a real fixed point. As can be read off from Tab. II,  $\mathcal{I}$  and  $\mathcal{J}$  are critical fixed points with exactly one RG relevant direction for  $N_f$  above (but close to)  $N_f^{(1)}$  and exchange one stable direction with the third fixed point  $\mathcal{H}$ , which becomes critical only for  $N_f < N_f^{(1)}$ . Thus, at this *triple collision* both phenomena, merging and disappearing into complex plane as well as exchanging roles with respect to RG stability, are realized. The analogous behavior can be simultaneously observed for the second triple of fixed points with 2 respectively 3 relevant directions (these fixed points are not among those explicitly detailed in Tab. II). To our knowledge, the present  $U(N_f) \times U(N_f)$  system represents the first-known example displaying such triple collision.

The second value of  $N_f$  at which we find degenerate fixed-point solutions is

$$N_f^{(2)} = 6. \quad (66)$$

Here, two simultaneous collisions of the first kind (à la Wilson-Fisher in  $d \nearrow 4$ ) occur: For  $N_f \nearrow N_f^{(2)}$  we find that fixed point  $\mathcal{I}$  collides with the Thirring fixed point

$$\mathcal{C} : (g_S^*, g_P^*, g_V^*, g_T^*) = (0, c_1, c_2, 0), \quad (67)$$

with the coordinates  $c_{1,2} \equiv c_{1,2}(N_f) < 0$ , i.e.,  $i_1 = 0$ ,  $i_2 = c_1$ ,  $i_3 = c_2$ , and  $i_4 = 0$  for  $N_f = N_f^{(2)}$ . The Thirring fixed point, which lies in the higher-symmetric  $U(2N_f)$  subspace, has two relevant directions for  $2 \leq N_f < 6$  and exchanges roles with respect to stability of one of its RG relevant directions with fixed point  $\mathcal{I}$  for  $N_f = N_f^{(2)} = 6$ . Only for  $N_f > N_f^{(2)}$  (and  $N_f = 1$ , see above) the Thirring fixed point is thus a critical fixed point, with  $\mathcal{I}$  being critical below (but close to)  $N_f^{(2)}$  and developing a second relevant direction above  $N_f^{(2)}$ . The same behavior can be observed for the fixed point  $\mathcal{J}$  that collides and exchanges its role with respect to RG stability with the fixed point

$$\mathcal{K} : (g_S^*, g_P^*, g_V^*, g_T^*) = (0, c_1, 0, c_2), \quad (68)$$

simultaneously at  $N_f = N_f^{(2)}$ . We note that the simultaneous collision of *two* pairs of fixed points, just as the fact that both triples collide at the same  $N_f^{(1)}$  as discussed above, is a consequence of the invariance of the flow equations [Eqs. (44)–(47)] under the exchange of  $g_V$  and  $g_T$ , which ensures that if  $(g_S^*, g_P^*, g_V^*, g_T^*)$  solves the fixed-point equations, so does  $(g_S^*, g_P^*, g_T^*, g_V^*)$ , and both (not necessarily distinct) fixed points have the same critical exponents. We discuss the stability of the  $U(2N_f)$  subspace in the vicinity of the Thirring fixed point in detail in subsection VIII C.

Just as in the one-flavor case, the two (irreducible and reducible) Gross-Neveu axes given by pure  $(\bar{\psi}\gamma_{35}\psi)^2$  and  $(\bar{\psi}\psi)^2$  interactions, respectively, each define a RG invariant one-dimensional subspace, since they contain the non-Gaussian fixed points

$$\mathcal{A}: (g_S^*, g_P^*, g_V^*, g_T^*) = (0, a, 0, 0) \quad (69)$$

and

$$\mathcal{D}: (g_S^*, g_P^*, g_V^*, g_T^*) = (a, 0, 0, 0), \quad (70)$$

respectively, where  $a \equiv a(N_f) > 0$ . Now, for  $N_f \geq 2$ , both  $\mathcal{A}$  and  $\mathcal{D}$  are critical fixed points with exactly one relevant direction—in contrast to the one-flavor case, where  $\mathcal{D}$  exhibited two relevant directions. All other fixed points, not listed in Tab. II, have two or more relevant directions for all  $N_f \geq 2$ .

Let us briefly summarize the results we have obtained so far (cf. Tab. II): For  $2 \leq N_f < N_f^{(1)} \simeq 3.76$  there are three critical fixed points  $\mathcal{A}$ ,  $\mathcal{D}$ , and  $\mathcal{H}$ . A pair  $\mathcal{I}$  and  $\mathcal{J}$  emerges from the complex plane at the location of  $\mathcal{H}$  when  $N_f = N_f^{(1)}$ . For  $N_f^{(1)} < N_f < N_f^{(2)}$  there are then four critical points  $\mathcal{A}$ ,  $\mathcal{D}$ ,  $\mathcal{I}$ , and  $\mathcal{J}$ . At  $N_f^{(2)} = 6$  the fixed points  $\mathcal{I}$  and  $\mathcal{J}$  collide with  $\mathcal{C}$  and  $\mathcal{K}$ , respectively. For  $N_f > N_f^{(2)}$  we still have four critical fixed points, which however are now  $\mathcal{A}$ ,  $\mathcal{D}$ ,  $\mathcal{C}$ , and  $\mathcal{K}$ .

## B. RG invariant subspaces

The fixed-point structure can further be elucidated by considering the flow in subspaces that are closed under the action of the RG. Besides the RG-closed one-dimensional lines connecting each non-Gaussian fixed point with the Gaussian fixed point, we find three RG invariant two-dimensional planes, each consisting of three non-Gaussian fixed points and the Gaussian fixed point. These are

- (1) the Thirring subspace, defined by

$$(g_S, g_P, g_V, g_T) = (0, g_P, g_V, 0),$$

consisting of the fixed point  $\mathcal{A}$ , i.e., the non-Gaussian fixed point of the Gross-Neveu model in the irreducible representation of the Clifford algebra,  $\mathcal{B}$  at  $g^* = (0, b_1, b_2, 0)$ , and the Thirring fixed point  $\mathcal{C}$ ,

- (2) the Thirring subspace's equivalent for  $g_V \leftrightarrow g_T$ , given by

$$(g_S, g_P, g_V, g_T) = (0, g_P, 0, g_T),$$

consisting of fixed point  $\mathcal{A}$ , the Thirring fixed point's equivalent fixed point  $\mathcal{K}$ , and a not further specified fixed point at  $g^* = (0, b_1, 0, b_2)$  (equivalent of  $\mathcal{B}$ ),

- (3) the Gross-Neveu subspace, with

$$(g_S, g_P, g_V, g_T) = (g_S, g_P, 0, 0),$$

consisting of the fixed points  $\mathcal{A}$  and  $\mathcal{D}$  of the irreducible and reducible Gross-Neveu models, as well as the fixed point  $\mathcal{G}$  at  $g_S^* = g_P^*$ ,

- (4) a not further specified subspace, in which

$$(g_S, g_P, g_V, g_T) = (g_S, g_S, g_V, g_V),$$

consisting of the fixed points  $\mathcal{H}$ ,  $\mathcal{L}$ , and  $\mathcal{G}$ . By means of the Fierz identities (18)–(21), we can rewrite the interactions in this subspace as a linear combination of the two Fierz-transformed (“dual”) interactions  $(S^D)^2$  and  $(V^D)^2$ :

$$\begin{aligned} & g_S(S)^2 + g_S(P)^2 + g_V(V)^2 + g_V(T)^2 \\ &= -\frac{1}{2}(g_S + 3g_V)(S^D)^2 - \frac{1}{2}(g_S - g_V)(V^D)^2. \end{aligned} \quad (71)$$

Subspaces (1) and (3) can be associated with the corresponding one-flavor subspaces studied above. The RG flow in subspace (2) is completely equivalent to the Thirring subspace (1). The invariant  $(S^D)^2$ – $(V^D)^2$  subspace (4) is new. We have depicted the RG flow within the invariant planes (1), (3), and (4) in Fig. 2 for the case of  $N_f = 2$ . While the flow in directions orthogonal to an invariant plane vanishes *per definitionem*, we emphasize that the answer to the interesting question whether or not small perturbations out of the plane are RG relevant depends on the location considered on the plane, as well as on the flavor number  $N_f$ . For instance, in the vicinity of the fixed point  $\mathcal{H}$  (cf. right panel of Fig. 2), which is a critical fixed point for  $2 \leq N_f < N_f^{(1)}$ , small perturbations orthogonal to the  $(S^D)^2$ – $(V^D)^2$ -plane are RG irrelevant (relevant) for  $2 \leq N_f < N_f^{(1)}$  ( $N_f > N_f^{(1)}$ ). By contrast, near the fixed point  $\mathcal{L}$ , which for all  $N_f \geq 2$  has four relevant directions, perturbations are always relevant. For all  $N_f \geq 2$  and in the vicinity of all four fixed points  $\mathcal{O}$ ,  $\mathcal{A}$ ,  $\mathcal{D}$ , and  $\mathcal{G}$ , the Gross-Neveu plane  $g_V = g_T = 0$  (middle panel of Fig. 2) is stable under small perturbations out of this plane. (Note that this was *not* the case near the fixed point  $\mathcal{D}$  for  $N_f = 1$ , see above.) In the following we discuss in detail the stability of the higher-symmetric  $U(2N_f)$  subspace against symmetry-breaking perturbations in the vicinity of the Thirring fixed point  $\mathcal{C}$  (cf. left panel of Fig. 2), as this question has been under some debate in the past [3, 16, 17, 19].



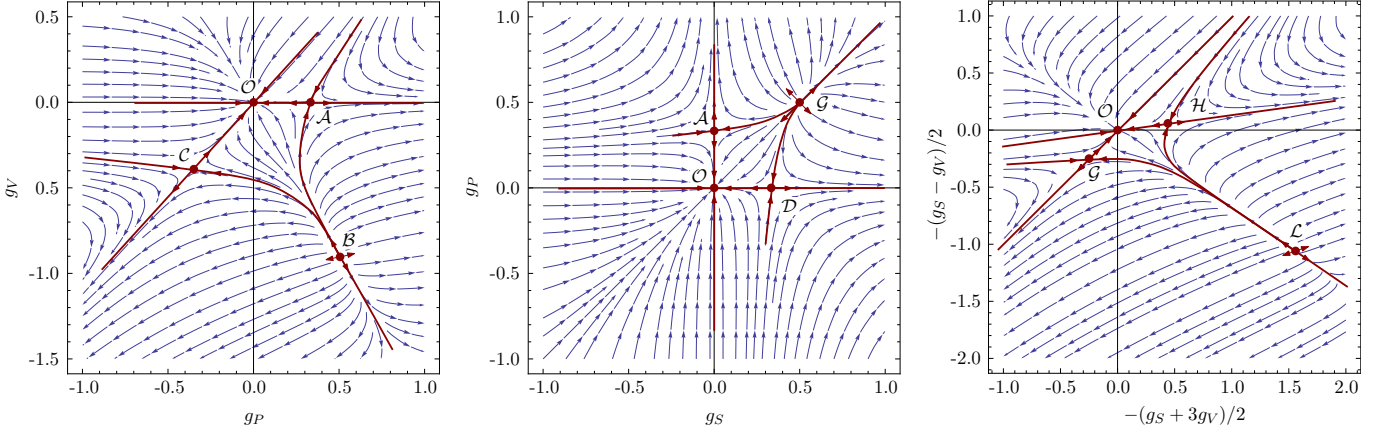


FIG. 2. Flow in RG invariant planes for  $N_f = 2$ . Arrows point towards the IR. Left: Thirring subspace with higher  $U(2N_f)$  symmetry for  $g_S = g_T = 0$  [15]. Middle: Gross-Neveu subspace for  $g_V = g_T = 0$ . Right: subspace defined by the Fierz-transformed interactions  $(S^D)^2$  and  $(V^D)^2$ , given by  $g_S = g_P$  and  $g_V = g_T$  and parametrized by the couplings  $g_S^D \equiv -\frac{1}{2}(g_S + 3g_V)$  (horizontal axis) and  $g_V^D \equiv -\frac{1}{2}(g_S - g_V)$  (vertical axis).

### C. (In-)stability of Thirring subspace against symmetry-breaking perturbations

The large- $N_f$  analysis [14] shows the nonperturbative renormalizability of the three-dimensional Thirring model with a single interaction parameter  $g_V$ , which is equivalent to saying that at large  $N_f$  the Thirring fixed point is critical, at least within the  $U(2N_f)$  subspace. In the context of the Thirring model's lattice version, at large  $N_f$  it has been shown that small perturbations that break the  $U(2N_f)$  symmetry are RG irrelevant and the  $U(2N_f)$  is IR attractive, at least in the vicinity of the Thirring fixed point [16]. Our RG analysis is consistent with the large- $N_f$  behavior, but shows that the IR attractiveness of the Thirring fixed point does *not* reach all the way down to  $N_f = 2$ . Instead, there exists a “critical” number of flavors  $N_f^{(2)}$  below which the Thirring subspace becomes IR repulsive and the Thirring fixed point develops a second relevant direction. Below  $N_f^{(2)}$ ,  $U(2N_f)$ -breaking perturbations are RG relevant and drive the flow to a different fixed point with a lower symmetry.

This is visualized in Fig. 3, which shows the RG flow in the plane spanned by the two most relevant directions at the Thirring fixed point  $\mathcal{C}$ , for different number of flavors  $N_f$ . Depending on where one starts the RG flow in the vicinity of the Thirring fixed point one finds a finite region of starting values for which the couplings flow to zero (Gaussian fixed point). Outside this region there is a runaway flow. At the boundaries of this region the flow runs into a critical fixed point with exactly one relevant direction. Only for  $N_f > N_f^{(2)} = 6$  is this the Thirring fixed point itself. Below  $N_f^{(2)}$ , these are either the fixed points  $\mathcal{H}$  and  $\mathcal{D}$  for  $2 \leq N_f < N_f^{(1)}$  or the fixed points  $\mathcal{I}$  and  $\mathcal{D}$  for  $N_f^{(1)} < N_f < N_f^{(2)}$ . Directly at  $N_f^{(2)}$  the Thirring fixed point merges with  $\mathcal{I}$ .

We emphasize that  $N_f^{(2)}$  should *not* be confused with the chiral-critical flavor number  $N_{f,c}^X$  below which the three-dimensional Thirring model is expected to allow chiral symmetry breaking [3, 15, 17]. We believe that these two phenomena are unrelated and the respective critical flavor numbers will most likely *not* coincide.

## IX. PROSPECTS ON LONG-RANGE PHYSICS

Within the present fermionic truncation of the effective average action it is generically hard to associate a given critical fixed point with a specific symmetry-breaking pattern and corresponding continuous phase transition. Moreover, the general structure of the beta functions as discussed in Sec. VI renders the single positive critical exponent  $\Theta = d - 2$ , corresponding to a correlation-length exponent of the associated phase transition of  $\nu = 1/\Theta = 1/(d - 2)$ , independent of  $N_f$ . While indeed many fermionic universality classes in  $d = 3$ , e.g., of the Gross-Neveu-type [5–12, 23, 63, 64] seem to point to  $\nu \approx 1$  ( $\pm 20\%$ ), the insensitivity of the fermionic-truncation prediction to the specific transition clearly calls for more elaborate techniques to investigate the RG flow in the vicinity of a given fixed point. Within the functional RG, this is, for instance, possible by suitable partial or dynamical bosonization techniques [43, 44]. The nature of the interacting phases expected at large coupling can also be investigated by computing the flow of the order-parameter susceptibilities [65], or the flows of full potentials for fermion bilinears [55]. Here, we content ourselves with an outlook on *possible* symmetry breakings associated with the critical fixed points we have found and leave a more detailed analysis for future work.

Some of the fixed points considered in the present work have been discussed earlier. Let us start with the

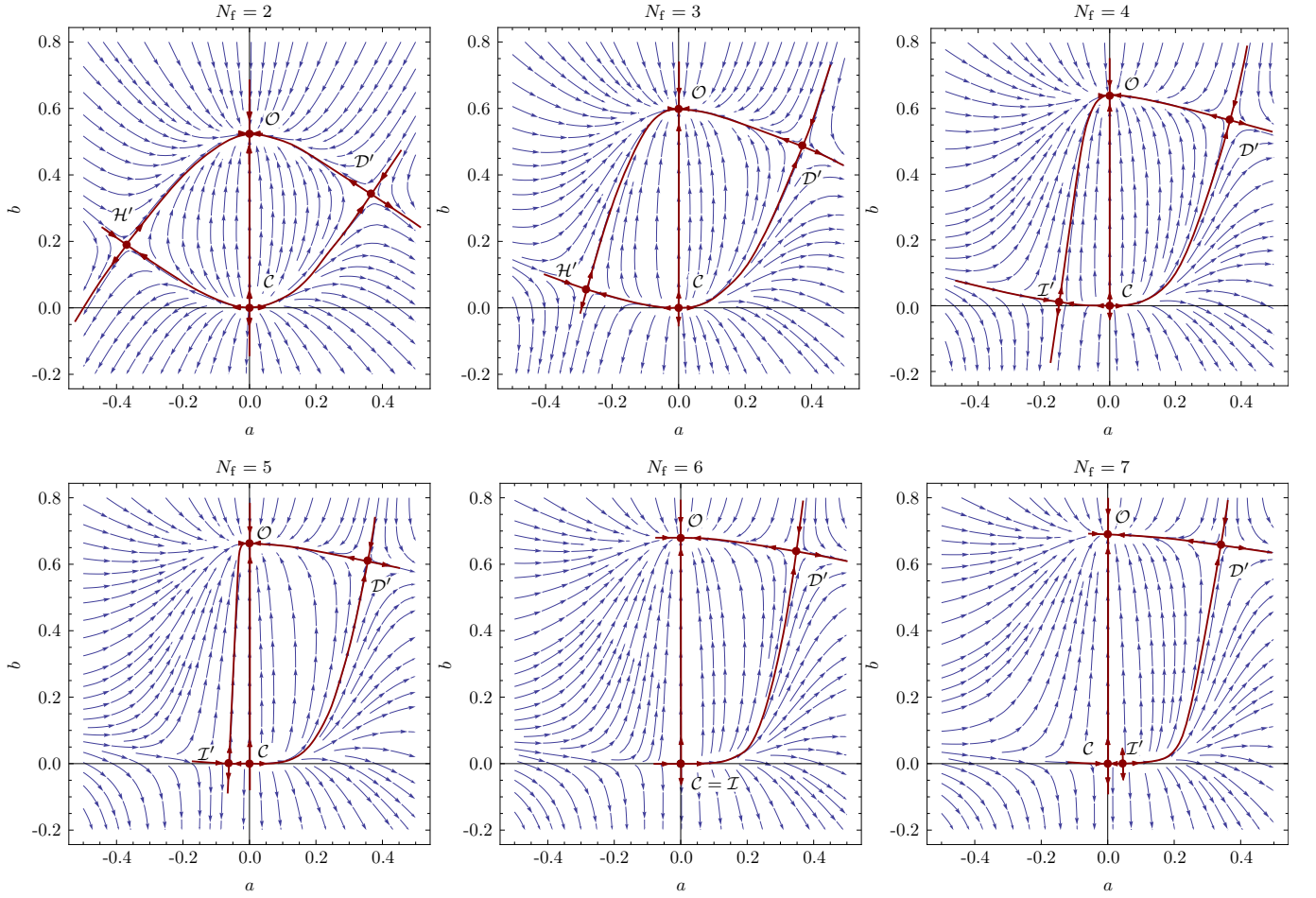


FIG. 3. RG flow in the (RG noninvariant!) plane spanned by the Thirring fixed point  $\mathcal{C}$  and the corresponding eigendirections  $\mathbf{v}_{1,2}$  with the largest corresponding exponents  $\Theta_{1,2}$ . This plane is given by  $\mathbf{g} = \mathbf{g}_\mathcal{C}^* + a\mathbf{v}_1 + b\mathbf{v}_2$  with coordinates  $a, b \in \mathbb{R}$  and basis vectors  $\mathbf{v}_2 \propto \mathbf{g}_\mathcal{C}^*$  (corresponding exponent  $\Theta_2 = 1$ ) and  $\mathbf{v}_1$ , the latter being the second relevant direction for  $N_f < 6$ , becoming marginal for  $N_f = 6$ , and eventually irrelevant for  $N_f > 6$ . The vertical axis  $a = 0$  thus represents the Thirring subspace, while finite  $a \neq 0$  corresponds to  $U(2N_f)$ -symmetry-breaking perturbations. By varying the starting points of the RG flow in the vicinity of  $\mathcal{C}$  there is a certain finite region for which the flow runs into the Gaussian fixed point  $\mathcal{O}$ . Outside this region we find a runaway flow. At the boundaries the flow runs into a critical fixed point with exactly one relevant direction. This is the fixed point  $\mathcal{H}$  ( $\mathcal{D}$ ) if  $a < 0$  ( $a > 0$ ) for  $2 \leq N_f < N_f^{(1)}$  (top left and middle), the fixed point  $\mathcal{I}$  ( $\mathcal{D}$ ) if  $a < 0$  ( $a > 0$ ) for  $N_f^{(1)} < N_f < N_f^{(2)} = 6$  (top right and bottom left), and eventually the Thirring fixed  $\mathcal{C}$  itself (independent of  $|a| \ll 1$ ) for  $N_f > N_f^{(2)}$ , which only then becomes critical (bottom right). Directly at  $N_f = 6$  the Thirring fixed point merges with fixed point  $\mathcal{I}$  (bottom middle). In general,  $\mathcal{D}$ ,  $\mathcal{H}$ , and  $\mathcal{I}$  do not lie in the present plane—however, we can observe their “projections” onto this plane at the points where the flow in directions parallel to this plane vanishes. These projections are marked by  $\mathcal{D}'$ ,  $\mathcal{H}'$ , and  $\mathcal{I}'$ .

Thirring fixed point  $\mathcal{C}$ . In the  $U(2N_f)$  subspace, the Thirring fixed point is always a critical fixed point [15]. However, only for  $N_f = 1$  and  $N_f > N_f^{(2)} = 6 + \mathcal{O}(d-2)$  symmetry-breaking perturbations are RG irrelevant. For  $N_f = 1$  the three-dimensional Thirring model is expected to exhibit a spontaneous breaking of the “chiral”  $U(2N_f)$  symmetry at large coupling  $-g_V > -g_{V,c} \simeq -c_2(N_f) > 0$  with order parameter  $\langle \bar{\psi}\psi \rangle$ . The critical behavior has been discussed within a functional RG approach [20], yielding the following predictions for correlation-length critical exponent  $\nu$  and order-parameter anomalous di-

mension  $\eta$

$$\mathcal{C}(N_f = 1) : \quad \nu \approx 1.9 \quad \eta \approx 1.0. \quad (72)$$

For  $N_f > 6$ , where the Thirring fixed point again becomes critical also in the presence of  $U(2N_f)$ -breaking interactions, the ordering presumably no longer breaks the chiral symmetry [3], but the exact type of ordering is unknown. The ordering is also not clearly identifiable for the Thirring fixed point’s equivalent for  $g_V \leftrightarrow g_T$  for  $N_f > 6$ , i.e., the fixed point  $\mathcal{K}$ .

By contrast, the fixed point and the associated critical behavior of the three-dimensional Gross-Neveu model

in the irreducible representation of the Clifford algebra ( $\mathcal{A}$  in our notation) is fairly well known [5–10]. As expected from the location of the fixed point on the  $(\bar{\psi}\gamma_{35}\psi)^2 = (\bar{\chi}\chi)^2$  axis, a nonvanishing order parameter  $\langle\bar{\psi}\gamma_{35}\psi\rangle = \langle\bar{\chi}\chi\rangle \neq 0$  occurs for large coupling  $g_P > g_P^*|_{\mathcal{A}} \equiv a$  and thus breaks the parity symmetry while leaving the  $U(2N_f)$  symmetry intact. Here again,  $\psi$  corresponds to  $N_f$  flavors of reducible four-component Dirac spinors and  $\chi$  to the corresponding  $2N_f$  two-component (Weyl) spinors [see Eq. (17)]. In the system of interacting fermions on graphene’s honeycomb lattice the phase transition into the quantum anomalous Hall state that is predicted for large next-to-nearest neighbor interactions (at least for  $N_f = 1$  [28]) is expected to be governed by fixed point  $\mathcal{A}$  [22]. The critical behavior (for the example of  $N_f = 2$ ) is determined by the exponents

$$\mathcal{A}(N_f = 2) : \quad \nu \approx 0.95 \dots 1.04, \quad \eta \approx 0.70 \dots 0.78, \quad (73)$$

where the ranges indicate the different predictions obtained by  $(4 - \epsilon)$  expansion [6], large- $N_f$  expansion [8], Monte-Carlo simulations [9], and functional RG [10]. A recent overview of the literature results can be found in Ref. [23].

The fixed point  $\mathcal{D}$  of the Gross-Neveu model in the reducible representation has been considered in Refs. [12, 22, 63, 64].  $\mathcal{D}$  determines the critical behavior of the discrete-chiral-symmetry breaking with order parameter  $\langle\bar{\psi}\psi\rangle$  which becomes finite for  $g_S > a$ . On the honeycomb lattice, it has been ascribed to the transition into the charge density wave phase that is expected for large nearest-neighbor interaction [21, 22]. The critical behavior for  $N_f \geq 2$  coincides with the one in the irreducible Gross-Neveu model [Eq. (73)], at least to the order that the exponents have been computed so far [10, 12, 63, 64].<sup>4</sup> However, we would like to emphasize again that  $\mathcal{D}$  develops *two* relevant directions in the one-flavor case. There is therefore a fundamental difference between the irreducible and reducible Gross-Neveu models for  $N_f = 1$ : As long as only one microscopic parameter of a model, e.g., for spinless fermions on the honeycomb lattice, is tuned, only  $\mathcal{A}$  describes a second order phase transition, and it can be driven by increasing  $g_P$  for small  $g_S$  and  $g_V$ . By increasing the microscopic coupling  $g_S$  and keeping  $g_P$  and  $g_V$  small and, say, positive we might find a phase transition, which, however, is *not* governed by fixed point  $\mathcal{D}$  but instead by the critical fixed point  $\mathcal{E}$ .

Within the present approximation it is hard to decide which order parameter is induced at fixed point  $\mathcal{E}$ . A simplistic approach often applied [40] is given by keeping

track of the “amount of divergence” of the various condensation channels, in order to determine in which channel the couplings diverges “first”. In our one-loop flow, the line connecting  $\mathcal{E}$  and the Gaussian fixed point  $\mathcal{O}$  is a RG attractive one-dimensional subspace for  $N_f = 1$  and positive couplings. E.g., if we start the flow in the UV near the reducible-Gross-Neveu axis at  $\mathbf{g} = (g_S, g_V, g_P)$  with  $0 < g_V, g_P \ll g_S$  and  $g_S$  above but close to the critical  $g_S^* = 1/2$ , the couplings will always run to the  $\mathcal{O}\mathcal{E}$  axis before they eventually diverge at a finite RG scale  $t_0$ . From the values of the fixed-point couplings [Eq. (61)] we find the following ordering of couplings for  $t \rightarrow t_0$

$$\frac{g_S}{g_V} \rightarrow 3, \quad \frac{g_P}{g_V} \rightarrow 1.38, \quad g_S, g_P, g_V \rightarrow \infty, \quad (74)$$

i.e., the coupling  $g_S$  diverges “fastest”. From this simplistic analysis one might thus speculate that fixed point  $\mathcal{E}$  governs a condensation in the  $(S)^2$  channel with order parameter  $\langle\bar{\psi}\psi\rangle$ . That is, with increase of the microscopic coupling  $g_S$  the Gross-Neveu model with reducible, four-component, Dirac spinors should exhibit a continuous phase transition beyond which the discrete  $\mathbb{Z}_2$  chiral symmetry is spontaneously broken, a prediction that is consistent with the mean-field theory for a system with large  $g_S(\bar{\psi}\psi)^2$  interaction and  $g_S > 0$  [12]. This, of course, requires confirmation beyond the present analysis. In any case, however, we believe there is no reason to expect the critical fixed point  $\mathcal{E}$  to exhibit the same critical behavior as fixed point  $\mathcal{A}$ . In our approximation, the largest critical exponent is  $\Theta = (d-2) + \mathcal{O}((d-2)^2)$  at any critical fixed point. Yet, already the second-largest exponents that can be associated to the (universal) corrections to scaling do not coincide: at fixed point  $\mathcal{A}$  we obtain

$$\mathcal{A}(N_f = 1) : \quad \omega = 2(d-2) + \mathcal{O}((d-2)^2), \quad (75)$$

whereas for fixed point  $\mathcal{E}$  we get

$$\mathcal{E}(N_f = 1) : \quad \omega = (3 - \sqrt{5})(d-2) + \mathcal{O}((d-2)^2). \quad (76)$$

Beyond our approximation, one would expect already the leading exponents, for instance  $\nu$  or  $\eta$ , to receive different corrections at the two inequivalent fixed points. If true, the irreducible and reducible Gross-Neveu models as defined by fixed points  $\mathcal{A}$  and  $\mathcal{E}$  thus represent an example of two three-dimensional fermion systems which both show spontaneous breaking of  $\mathbb{Z}_2$  symmetry, but differ in their corresponding critical behavior. This touches a general issue on universality in fermionic systems: What are the defining properties that determine a specific universality class? Our results suggest that in fermionic systems the symmetry of the order parameter and the dimension and field content of the given system does not yet uniquely define the critical behavior. Instead, additional “spectator symmetries” that do not take part in the symmetry breaking pattern might also

<sup>4</sup> In fact, the RG flows of the reducible Gross-Neveu model studied in [12, 63, 64] are in principle identical to those of the irreducible Gross-Neveu model [10] within the truncations focusing on the dynamics of the bosonic order parameter considered so far. In view of our results for the larger theory space, the literature results for the reducible case for  $N_f = 1$  should rather be considered as applying more appropriately to the irreducible case.

play a decisive role. In our case, this is the  $U(2N_f)$  versus  $\mathbb{Z}_2 \times U(N_f) \times U(N_f)$  symmetry that discriminates between the irreducible and reducible Gross-Neveu models. We believe that these general questions on universality are an interesting direction for future research, and the models presented here constitute a suitable playground to study them.

The fixed point  $\mathcal{H}$  that becomes critical for  $2 \leq N_f < N_f^{(1)} \approx 3.8$  has the interesting property that for  $N_f = 3.5$  it is located exactly on the axis at which all four couplings coincide,  $g_S = g_P = g_V = g_T < 0$ . By means of the Fierz identities the system on this axis can be written with pure  $(S^D)^2 = (\bar{\psi}^a \psi^b)^2 + (\bar{\psi}^a \gamma_{35} \psi^b)^2$  interaction [cf. Eq. (71)], and positive coupling  $g_S^D = -2g_S > 0$ . One would thus expect the  $(S^D)^2$  condensation channel to become critical at this fixed point, and this should remain true also for other values of  $N_f$  not too far from  $N_f = 3.5$ . However, the critical behavior of this system is unknown to us.

We are also not able to make any substantial comment on the IR behaviors of the fixed points  $\mathcal{I}$  and  $\mathcal{J}$ . Both are critical fixed points for  $N_f^{(1)} < N_f < N_f^{(2)}$ , but cannot be associated to one of the possible condensation channels in an obvious way.

## X. CONCLUSIONS

Our study of a general class of relativistic fermion theories in  $2 < d < 4$  space-time dimensions with continuous chiral  $U(N_f) \times U(N_f)$  symmetry has revealed a network of RG fixed points. In a unified framework, our description includes a number of well-studied models such as various versions of the Gross-Neveu model and the Thirring model. If persistent also beyond our simple pointlike approximation, each fixed point defines its own universality class thus facilitating both different continuum limits and corresponding microscopic “theories” as well as a diversity of possible long-range phenomena.

We pay particular attention to those fixed points with only one RG relevant direction. These are candidates for critical points of a second-order quantum phase transition. It is one of our main results that the nature of these critical fixed points in the present model depends on the number of flavors. While the fixed point  $\mathcal{A}$  corresponding to the irreducible Gross-Neveu model is a critical fixed point for any number of flavors, the fixed point  $\mathcal{D}$  for the reducible Gross-Neveu model is only a critical fixed point for  $N_f \geq 2$  whereas it is not for  $N_f = 1$ . Most interestingly, the Thirring fixed point  $\mathcal{C}$  is a critical fixed point for  $N_f = 1$  and for  $N_f > 6$ . While all these results are in agreement with the corresponding large- $N_f$  analyses of these theories, they indicate that both the microscopic as well as the long-range behavior of these systems can change drastically as a function of  $N_f$ .

Within the limit of pointlike interactions, we have proven a number of properties of invariant subspaces of the RG flow. While subspaces of higher symmetry are always guaranteed to be invariant subspaces, the structure

of our RG flows allows to define further criteria that are not necessarily related to higher symmetries. Subspaces with higher symmetry can lead to the phenomenon of emergent symmetry in the long-range physics if this subspace is not only invariant but also RG attractive towards the IR. Again we found that the emergence of higher symmetry can be a flavor-number dependent property. For instance, the Thirring subspace of higher  $U(2N_f)$  symmetry is attractive only for  $N_f > N_f^{(2)}$  and  $N_f = 1$  in the vicinity of the Thirring fixed point  $\mathcal{C}$ .

As our approach is equivalent to the one-loop  $(2 + \epsilon)$ -expansion, at the very least it establishes the *existence* of  $N_f^{(2)}$  and its accompanying qualitatively different behavior for  $N_f < N_f^{(2)}$ . One should expect that our estimate for its value in  $d = 3$  within the present simple truncation is subject to quantitative improvement beyond the pointlike limit. Nevertheless, we find it interesting that our one-loop result  $N_f^{(2)} = 6 + \mathcal{O}(d - 2)$  appears to be near the number  $N_{f,c}^{\text{lattice}} = 6.6(1)$  at which the IR observables in the lattice simulations show an abrupt change [17]. These simulations employ a lattice formulation of fermions (staggered fermions) that microscopically breaks parts of the  $U(2N_f)$  symmetry. It is therefore of decisive importance for the interpretation of the simulation results whether these perturbations are relevant in the RG sense or not. Our analysis suggests that the answer to this question might in fact depend on the flavor number, with a large- $N_f$  regime in which perturbations are irrelevant and an intermediate- $N_f$  regime in which perturbations become relevant. The boundary is given by  $N_f^{(2)}$ . It should be interesting for future work to establish whether indeed the abrupt change found at  $N_{f,c}^{\text{lattice}}$  in the simulations signals—instead of an upper bound for chiral symmetry breaking—a change in the number of relevant directions at the UV Thirring fixed point. Such an interpretation would reconcile the seeming disagreement between  $N_{f,c}^{\text{lattice}}$  and the majority of the analytical estimates for  $N_{f,c}^X$ , which appear to be significantly lower than  $N_{f,c}^{\text{lattice}}$  [3, 66]. It could potentially also resolve the contradiction between the critical behaviors found in the continuum Thirring model and its lattice version, as discussed in [3].

For  $N_f = 1$ , our theory space describes the system of interacting spinless fermions on the honeycomb lattice, a simple model for graphene. This model has thoroughly been investigated previously [22, 28, 67, 68]. Within our RG approach, we rediscover the simple mean-field phase diagram which exhibits besides the semimetallic phase at weak coupling two gapped phases, which are approached by tuning nearest- and next-nearest-neighbor interactions, respectively, above a strong-coupling threshold. The appropriate order parameters are  $\langle \bar{\psi} \psi \rangle$  (charge density wave phase) and  $\langle \bar{\psi} \gamma_{35} \psi \rangle$  (quantum anomalous Hall phase), respectively, each indicating a spontaneous breaking of a  $\mathbb{Z}_2$  symmetry. We can associate these second-order quantum phase transitions with their corresponding critical fixed points. The transition into the



quantum anomalous Hall phase is determined by the irreducible-Gross-Neveu fixed point  $\mathcal{A}$ , located on the axis with pure  $(\bar{\psi}\gamma_{35}\psi)^2$ , as one would naively also expect from mean-field theory. A similar expectation, however, could fail in the case of the charge density wave transition. Our results indicate that this transition is *not* to be associated with the reducible-Gross-Neveu fixed point  $\mathcal{D}$ , which is located on the axis with pure  $(\bar{\psi}\psi)^2$  interaction and for  $N_f = 1$  has two RG relevant directions. Instead, the charge density wave transition may be governed by fixed point  $\mathcal{E}$ , at which all short-range interactions become finite.

Our results also provide evidence for a new feature of universality: whereas universality is conventionally considered as being governed by the symmetry of the order parameter, the dimensionality and the number of long-range interactions, our theory space appears to contain a possible counter-example: for  $N_f = 1$ , there are two critical fixed points in the theory space ( $\mathcal{A}$  and  $\mathcal{E}$  in our notation) which are likely to be associated with a second-order phase transition involving the breaking of a  $\mathbb{Z}_2$  symmetry. In the irreducible Gross-Neveu model, the phase transition describes a breaking of parity symmetry, whereas the reducible Gross-Neveu model goes along with a breaking of a discrete chiral symmetry. In both cases, we have a  $\mathbb{Z}_2$  order parameter (real scalar field) and the same number of massless fermion modes near the phase transition. The main conceptual difference of the

two models lies in the additional spectator symmetries, which remain intact across the phase transition. More formally, the phase transition is related to two inequivalent RG fixed points lying in different invariant subspaces of the full theory space. Quantitatively, we observe that the corresponding critical exponents of the fixed points differ. Whereas our simple approximation yields differing exponents only for the subleading exponents, there is a priori no reason why an improved approximation should not lead to differences also for the leading relevant exponent. If this is the case for the phase diagram of spinless fermions on the honeycomb lattice, this would implicate that the two possible phase transitions lie in different universality classes. Even though both transitions would go along with the breaking of a discrete  $\mathbb{Z}_2$  symmetry, they should exhibit a different set of critical exponents. The recent advances in overcoming the sign problem in lattice Monte Carlo simulations [69] may allow to test this prediction in the near future.

## ACKNOWLEDGMENTS

We have substantially benefited from discussions with J. Borchardt and I. F. Herbut. This work has been supported by the DFG under Grants FOR 723, GRK 1523, and JA 2306/1-1, as well as the NSERC of Canada.

- 
- [1] I. Herbut, *A Modern Approach to Critical Phenomena* (Cambridge University Press, Cambridge, 2007).
  - [2] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); *ibid.* **124**, 246 (1961).
  - [3] L. Janssen and H. Gies, Phys. Rev. D **86**, 105007 (2012).
  - [4] J. Jaeckel, C. Wetterich, Phys. Rev. D **68**, 025020 (2003).
  - [5] B. Rosenstein, B. J. Warr, and S. H. Park, Phys. Rev. Lett. **62**, 1433 (1989).
  - [6] B. Rosenstein, H.-L. Yu, and A. Kovner, Phys. Lett. **B314**, 381 (1993).
  - [7] J. Gracey, Nucl. Phys. **B367**, 657 (1991); C. Luperini and P. Rossi, Ann. Phys. (NY) **212**, 371 (1991). A. Vasiliev and M. Vyazovsky, Theor. Math. Phys. **113**, 1277 (1997); J. Gracey, Nucl. Phys. **B802**, 330 (2008).
  - [8] A. Vasiliev, S. E. Derkachov, N. Kivel, and A. Stepanenko, Theor. Math. Phys. **94**, 127 (1993); J. A. Gracey, Int. J. Mod. Phys. A **09**, 727 (1994); M. Moshe and J. Zinn-Justin, Phys. Rep. **385**, 69 (2003).
  - [9] S. Hands, A. Kocic, and J.B. Kogut, Ann. Phys. **224**, 29 (1993); L. Karkkainen, R. Lacaze, P. Lacock, and B. Petersson, Nucl. Phys. **B415**, 781 (1994); E. Focht, J. Jersak, and J. Paul, Phys. Rev. D **53**, 4616 (1996); S. Christofi and C. Strouthos, J. High Energy Phys. **05** (2007) 088.
  - [10] L. Rosa, P. Vitale, and C. Wetterich, Phys. Rev. Lett. **86**, 958 (2001); F. Hofling, C. Nowak, and C. Wetterich, Phys. Rev. B **66**, 205111 (2002).
  - [11] H. Gies, L. Janssen, S. Rechenberger, and M. M. Scherer, Phys. Rev. D **81**, 025009 (2010).
  - [12] J. Braun, H. Gies, and D. D. Scherer, Phys. Rev. D **83**, 085012 (2011); J. Braun, J. Phys. G: Nucl. Part. Phys. **39**, 033001 (2012).
  - [13] S. Chandrasekharan and A. Li, Phys. Rev. D **85**, 091502(R) (2012); *ibid.* **88**, 021701(R) (2013).
  - [14] G. Parisi, Nucl. Phys. **B100**, 368 (1975); S. Hikami and T. Muta, Prog. Theor. Phys. **57**, 785 (1977); S. Hands, Phys. Rev. D **51**, 5816 (1995).
  - [15] H. Gies and L. Janssen, Phys. Rev. D **82**, 085018 (2010).
  - [16] L. Del Debbio, S. J. Hands, and J. C. Mehegan, Nucl. Phys. **B502**, 269 (1997); L. Del Debbio and S. J. Hands, *ibid.* **B552**, 339 (1999); S. Hands and B. Lucini, Phys. Lett. **B461**, 263 (1999).
  - [17] S. Christofi, S. Hands, and C. Strouthos, Phys. Rev. D **75**, 101701 (2007).
  - [18] S. Chandrasekharan, Phys. Rev. D **82**, 025007 (2010).
  - [19] S. Chandrasekharan and A. Li, Phys. Rev. Lett. **108**, 140404 (2012).
  - [20] D. Mesterhazy, J. Berges, and L. von Smekal, Phys. Rev. B **86**, 245431 (2012).
  - [21] I. F. Herbut, Phys. Rev. Lett. **97**, 146401 (2006); V. Juricic, I. F. Herbut, and G. W. Semenoff, Phys. Rev. B **80**, 081405(R) (2009); I. F. Herbut, V. Juricic, and O. Vafek, *ibid.* **80**, 075432 (2009);
  - [22] I. F. Herbut, V. Juricic, and B. Roy, Phys. Rev. B **79**, 085116 (2009).
  - [23] L. Janssen and I. F. Herbut, Phys. Rev. B **89**, 205403 (2014).
  - [24] S. Hands and C. Strouthos, Phys. Rev. B **78**, 165423

- (2008); W. Armour, S. Hands, and C. Strouthos, *ibid.* **81**, 125105 (2010).
- [25] A. Cortijo, F. Guinea, and M. A. H. Vozmediano, J. Phys. A **45**, 383001 (2012).
- [26] O. Vafeek and A. Vishwanath, Annu. Rev. Condens. Matter Phys. **5**, 83 (2014).
- [27] D. C. Elias *et al.*, Nat. Phys. **7**, 701 (2011); T. O. Wehling *et al.*, Phys. Rev. Lett. **106**, 236805 (2011).
- [28] S. Raghu, X.-L. Qi, C. Honerkamp, and S.-C. Zhang, Phys. Rev. Lett. **100**, 156401 (2008); C. Honerkamp, Phys. Rev. Lett. **100**, 146404 (2008); C. Weeks and M. Franz Phys. Rev. B **81**, 085105 (2010); A. Grushin, E. Castro, A. Cortijo, F. de Juan, M. Vozmediano, and B. Valenzuela, *ibid.* **87**, 085136 (2013).
- [29] B. Roy, V. Juricic, and I. F. Herbut, Phys. Rev. B **87**, 041401(R) (2013); B. Roy and I. F. Herbut, *ibid.* **82**, 035429 (2010).
- [30] S. Ryu, C. Mudry, C.-Y. Hou, and C. Chamon, Phys. Rev. B **80**, 205319 (2009).
- [31] S. Chadha and H. B. Nielsen, Nucl. Phys. **B217**, 125 (1983).
- [32] Petr Horava, Phys. Rev. D **79**, 084008 (2009).
- [33] H.-S. Goh, M. A. Luty, and S.-P. Ng, J. High Energy Phys. **01** (2005) 040.
- [34] S.-S. Lee, Phys. Rev. B **76**, 075103 (2007).
- [35] S.-K. Jian, Y.-F. Jiang, and H. Yao, Phys. Rev. Lett. **114**, 237001 (2015).
- [36] D. B. Kaplan, Nucl. Phys. B, Proc. Suppl. **129**, 109 (2004).
- [37] B. Roy, Phys. Rev. B **84**, 113404 (2011).
- [38] P. Calabrese, A. Pelissetto and E. Vicari, Phys. Rev. B **67**, 054505 (2003), A. Eichhorn, D. Mesterhazy and M. M. Scherer, Phys. Rev. E **88**, 042141 (2013), I. Boettcher, arXiv:1503.07817 [cond-mat.stat-mech].
- [39] C. Wetterich, Phys. Lett. **B301**, 90 (1993).
- [40] W. Metzner, M. Salmhofer, C. Honerkamp, V. Meden, and K. Schönhammer, Rev. Mod. Phys. **84**, 299 (2012); C. Platt, W. Hanke, and R. Thomale, Adv. Phys. **62**, 453 (2013).
- [41] K.-U. Giering and M. Salmhofer, Phys. Rev. B **86**, 245122 (2012); A. Eberlein and W. Metzner, *ibid.* **89**, 035126 (2014); S. A. Maier, A. Eberlein, and C. Honerkamp, *ibid.* **90**, 035140 (2014).
- [42] J. Lee, P. Strack, and S. Sachdev, Phys. Rev. B **87**, 045104 (2013).
- [43] H. Gies and C. Wetterich, Phys. Rev. D **65**, 065001 (2002); S. Floerchinger and C. Wetterich, Phys. Lett. **B680**, 371 (2009).
- [44] J. M. Pawłowski, Ann. Phys. (N.Y.) **322**, 2831 (2007).
- [45] O. Vafeek, Phys. Rev. B **82**, 205106 (2010).
- [46] I. F. Herbut and L. Janssen, Phys. Rev. Lett. **113**, 106401 (2014); L. Janssen and I. F. Herbut, arXiv:1503.04242 [cond-mat.str-el].
- [47] L. Savary, E.-G. Moon, and L. Balents, Phys. Rev. X **4**, 041027 (2014).
- [48] A. Jakovac and A. Patkos, Phys. Rev. D **88**, 065008 (2013); A. Jakovac, A. Patkos, P. Posfay, arXiv:1406.3195 [hep-th].
- [49] H. Gies and S. Lippoldt, Phys. Rev. D **89**, 064040 (2014); Phys. Lett. **B743**, 415 (2015); S. Lippoldt, arXiv:1502.05607 [hep-th].
- [50] W. Pauli, Ann. Inst. Henri Poincaré **6**, 109 (1936).
- [51] D. D. Scherer and H. Gies, Phys. Rev. B **85**, 195417 (2012).
- [52] J. Berges, N. Tetradis, and C. Wetterich, Phys. Rep. **363**, 223 (2002); B. Delamotte, Lect. Notes Phys. **852**, 49 (2012); H. Gies, *ibid.*, 287 (2012); J. Braun, J. Phys. G: Nucl. Part. Phys. **39**, 033001 (2012); A. Wipf, *ibid.* **100**, 257 (2013).
- [53] P. Kopietz, L. Bartosch, and F. Schütz, *Introduction to the Functional Renormalization Group* (Springer, Berlin, 2010).
- [54] J. Braun, L. Fister, J. M. Pawłowski and F. Rennecke, arXiv:1412.1045 [hep-ph].
- [55] K. I. Aoki, S. I. Kumamoto and D. Sato, Prog. Theor. Exp. Phys. **2014**, 043B05 (2014) [arXiv:1403.0174 [hep-th]]; arXiv:1304.3289 [hep-th].
- [56] K. Sun, H. Yao, E. Fradkin, and S. A. Kivelson, Phys. Rev. Lett. **103**, 046811 (2009).
- [57] S. Mallat, *A wavelet tour of signal processing: the sparse way* (Academic Press, 2008).
- [58] H. Gies, J. Jaeckel and C. Wetterich, Phys. Rev. D **69**, 105008 (2004).
- [59] L. Classen, I. F. Herbut, L. Janssen, and M. M. Scherer, arXiv:1503.05002 [cond-mat.str-el].
- [60] H. Gies and J. Jaeckel, Eur. Phys. J. C **46**, 433 (2006); D. B. Kaplan, J.-W. Lee, D. T. Son, and M. A. Stephanov, Phys. Rev. D **80**, 125005 (2009); J. Braun, C. S. Fischer and H. Gies, Phys. Rev. D **84**, 034045 (2011).
- [61] B. I. Halperin, T. C. Lubensky, and S.-K. Ma, Phys. Rev. Lett. **32**, 292 (1974); I. F. Herbut and Z. Tesanovic, *ibid.* **76**, 4588 (1996); *ibid.* **78**, 980 (1997).
- [62] K. Kaveh and I. F. Herbut, Phys. Rev. B **71**, 184519 (2005); J. Braun, H. Gies, L. Janssen, and D. Roscher, Phys. Rev. D **90**, 036002 (2014).
- [63] J. Borchardt and B. Knorr, arXiv:1502.07511 [hep-th].
- [64] G. P. Vacca and L. Zambelli, arXiv:1503.09136 [hep-th].
- [65] M. Salmhofer, C. Honerkamp, W. Metzner, and O. Lauscher, Progr. Theor. Phys. **112**, 943 (2004).
- [66] M. Gomes, R. S. Mendes, R. F. Ribeiro, and A. J. da Silva, Phys. Rev. D **43**, 3516 (1991); K. Kondo, Nucl. Phys. **B450**, 251 (1995); T. Itoh, Y. Kim, M. Sugiura, and K. Yamawaki, Prog. Theor. Phys. **93**, 417 (1995); M. Sugiura, Prog. Theor. Phys. **97**, 311 (1997).
- [67] L. Wang, P. Corboz, and M. Troyer, New J. Phys. **16**, 103008 (2014); Z.-X. Li, Y.-F. Jiang, H. Yao, arXiv:1408.2269 [cond-mat.str-el].
- [68] N. A. Garcia-Martinez, A. G. Grushin, T. Neupert, B. Valenzuela, and E. V. Castro, Phys. Rev. B **88**, 245123 (2013); M. Daghofer and M. Hohenadler, Phys. Rev. B **89**, 035103 (2014); T. Duric, N. Chancellor, and I. F. Herbut, Phys. Rev. B **89**, 165123 (2014); S. Capponi, A. M. Luchli, arXiv:1505.01676 [cond-mat.str-el]; J. Motruk, A. G. Grushin, F. de Juan, F. Pollmann, arXiv:1505.01674 [cond-mat.str-el].
- [69] E. F. Huffman and S. Chandrasekharan, Phys. Rev. B **89**, 111101(R) (2014).